Atoms and photons

Chapter 2

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Outline

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Interaction Hamiltonian

We consider a single electron atom (Hydrogen). The free Hamiltonian is:

$$H_0 = \frac{P^2}{2m} + qU(\mathbf{R}) \tag{1}$$

P and **R**: momentum and position operators. Eigenstates $H_0 |i\rangle = E_i |i\rangle$, ground state $|g\rangle$
Atom in a radiation field (potential vector $\mathbf{A}(\mathbf{r}, t)$, scalar potential $\mathbf{V}(\mathbf{r}, t)$):

$$H = \frac{1}{2m} (\mathbf{P} - q\mathbf{A}(\mathbf{R}, t))^2 + qU(\mathbf{R}) + qV(\mathbf{R}) \tag{2}$$

Note that $\mathbf{A}(\mathbf{R}, t)$ is an operator in the electron's Hilbert space.

Interaction Hamiltonian

Gauge choice

Gauge transformation

$$\begin{aligned} \mathbf{A}' &\rightarrow \mathbf{A} = \mathbf{A}' + \boldsymbol{\nabla}\chi(\mathbf{r}, t) \\ V' &\rightarrow V = V' - \frac{\partial \chi}{\partial t} \end{aligned} \tag{3}$$

where χ is an arbitrary function of space and time.

Coulomb gauge

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \tag{4}$$

Interaction Hamiltonian

Fourier space

Space-time Fourier transform

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{4\pi^2}\int \boldsymbol{\mathcal{A}}(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,d\mathbf{k}d\omega \tag{5}$$

Longitudinal and transverse potentials w.r.t. $\mathbf{k}$:

$$\boldsymbol{\mathcal{A}}(\mathbf{k},\omega) = \boldsymbol{\mathcal{A}}_{\parallel} + \boldsymbol{\mathcal{A}}_{\perp} \tag{6}$$

Hence:

$$\mathbf{A}(\mathbf{k},\omega) = \mathbf{A}_{\parallel} + \mathbf{A}_{\perp} \tag{7}$$

Space-time Fourier transform of $\boldsymbol{\nabla}\cdot\mathbf{A}$: $i\mathbf{k}\cdot\boldsymbol{\mathcal{A}}$. Coulomb:

$$\boldsymbol{\mathcal{A}}_{\parallel} = \mathbf{A}_{\parallel} = 0 \tag{8}$$

Interaction Hamiltonian

Fourier space

Same decomposition for fields. Transverse electric field, since divergence-free as **A** in the Coulomb gauge. With

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} V \tag{9}$$

and the fact that $\boldsymbol{\nabla} V$ is longitudinal (proportional to **k** in Fourier space)

$$\boldsymbol{\nabla} V = 0 \tag{10}$$

and (no physical effect of a constant potential)

$$V = 0 \tag{11}$$

Interaction Hamiltonian

$\mathbf{A}\cdot\mathbf{P}$ interaction

Expansion of $(\mathbf{P}-q\mathbf{A}(\mathbf{R},t))^2$ taking care of the commutation of $\mathbf{P}$ with $\mathbf{A}$. Noting:

$$[P_i, f(\mathbf{R})] = -i\hbar\frac{\partial f}{\partial R_i} \quad i\in\{x,y,z\} \tag{12}$$

$$\sum_i [P_i, A_i] = -i\hbar\sum_i \frac{\partial A_i}{\partial R_i} = -i\hbar\boldsymbol{\nabla}\cdot\mathbf{A} = 0 \tag{13}$$

$$\mathbf{P}\cdot\mathbf{A} = \sum_i P_i A_i = \sum_i A_i P_i = \mathbf{A}\cdot\mathbf{P} \tag{14}$$

And finally

$$H = \frac{P^2}{2m} + qU(\mathbf{R}) - \frac{q}{m}\mathbf{P}\cdot\mathbf{A} + \frac{q^2}{2m}\mathbf{A}\cdot\mathbf{A} \tag{15}$$

Weak fields (much lower than atomic field unit, 10^{11} V/m), $\mathbf{A}\cdot\mathbf{A}$ quadratic term negligible compared to first order contribution.

$$H = H_0 - \frac{q}{m}\mathbf{P}\cdot\mathbf{A}(\mathbf{R},t)\ . \tag{16}$$

Interaction Hamiltonian

$\mathbf{A} \cdot \mathbf{P}$ interaction: dipole approximation

- Radiation wavelength: about 1 μm
- Atomic size: about 100 pm
- Neglect spatial variation of the vector potential across atomic orbit: $\mathbf{A}(\mathbf{R}, t) = \mathbf{A}(0, t)$

$$H = H_0 - \frac{q}{m}\mathbf{P} \cdot \mathbf{A}(0, t) \ , \tag{17}$$

Useful, but not the intuitive form for the interaction of a dipole with a field.

Interaction Hamiltonian

$\mathbf{D} \cdot \mathbf{E}$ interaction

Cast the interaction Hamiltonian in the more familiar form $-\mathbf{d} \cdot \mathbf{E}$ (interaction energy of a dipole with a field, manifestly independent of the gauge choice). Two possible (and equivalent) approaches

1. The Göppert-Mayer transformation
2. Unitary transformation on the Hilbert space

Interaction Hamiltonian

The Göppert-Mayer transformation

Restart from full Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{P} - q\mathbf{A}(\mathbf{R}, t)\right)^2 + qU(\mathbf{R}) + qV(\mathbf{R}) \tag{18}$$

and perform dipole approximation first. For the vector potential

$$\mathbf{A}(\mathbf{R}, t) = \mathbf{A}(0, t) \tag{19}$$

and (keeping first order)

$$V = V(0, t) + \mathbf{R} \cdot \boldsymbol{\nabla} V(0, t) \tag{20}$$

The space-independent term in V has no effect

$$H = H_0 - \frac{q}{m}\mathbf{P} \cdot \mathbf{A}(0, t) + \mathbf{D} \cdot \boldsymbol{\nabla} V \tag{21}$$

with

$$\mathbf{D} = q\mathbf{R} \tag{22}$$

Interaction Hamiltonian

The Göppert-Mayer transformation

Perform a gauge transformation:

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi(\mathbf{r}, t) \\ V &\rightarrow V' = V - \frac{\partial \chi}{\partial t} \end{aligned} \tag{23}$$

and choose

$$\chi(\mathbf{r}, t) = -\mathbf{r} \cdot \mathbf{A}(0, t) \tag{24}$$

so that $\mathbf{A}'(0, t) = 0$. Then

$$V' = V + \mathbf{r} \cdot \frac{\partial \mathbf{A}(0, t)}{\partial t} \tag{25}$$

$$\boldsymbol{\nabla} V'(0) = \boldsymbol{\nabla} V(0) + \frac{\partial \mathbf{A}(0, t)}{\partial t} = -\mathbf{E}(0) \tag{26}$$

$$H = H_0 - \mathbf{D} \cdot \mathbf{E}(0) \tag{27}$$

Interaction Hamiltonian

Unitary transform approach

Restart from full Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{P} - q\mathbf{A}(\mathbf{R}, t)\right)^2 + qU(\mathbf{R}) + qV(\mathbf{R}) \tag{28}$$

Switch to Coulomb gauge (no V contribution left) and perform dipole approximation $\mathbf{A}(\mathbf{r}, t) = \mathbf{A}(0, t)$

$$H = \frac{1}{2m}\left(\mathbf{P} - q\mathbf{A}(0, t)\right)^2 + qU(\mathbf{R}) \tag{29}$$

Unitary transform $|\Psi\rangle \rightarrow \left|\widetilde{\Psi}\right\rangle = T\,|\Psi\rangle$ ($T^\dagger T = \mathbb{1}$). Transformed Hamiltonian

$$\widetilde{H} = THT^\dagger + i\hbar\frac{dT}{dt}T^\dagger \tag{30}$$

Interaction Hamiltonian

Unitary transform approach

Choose T as a time-dependent translation of the momentum:

$$T\mathbf{P}T^{\dagger} = \mathbf{P} + q\mathbf{A}(0,t) \tag{31}$$

$$T = e^{-\frac{i}{\hbar}q\mathbf{R}\cdot\mathbf{A}(0,t)} = e^{-\frac{i}{\hbar}\mathbf{D}\cdot\mathbf{A}(0,t)} \tag{32}$$

Hence

$$T\left(\mathbf{P} - q\mathbf{A}(0,t)\right)^2 T^{\dagger} = \mathbf{P}^2 \tag{33}$$

and

$$i\hbar\frac{dT}{dt}T^{\dagger} = \mathbf{D}\cdot\frac{d\mathbf{A}(0,T)}{dt} = -\mathbf{D}\cdot\mathbf{E}(0,t) \tag{34}$$

Finally,

$$\widetilde{H} = H_0 - \mathbf{D}\cdot\mathbf{E}(0) \tag{35}$$

Interaction Hamiltonian

Unitary transform approach

We get a transformed Hamiltonian in the $\mathbf{D} \cdot \mathbf{E}$ form, with a linear atom-field coupling.
We have not performed the weak field approximation to remove the $\mathbf{A} \cdot \mathbf{A}$ term in the Hamiltonian. Where is the magic?
The observables of the electron should be changed

$$O \rightarrow TOT^{\dagger} \tag{36}$$

and this change contains non linear terms in $\mathbf{A}$. It is only for weak fields that these terms can be neglected.

Non-resonant interaction

A simple situation:

- An atom initially in the ground state
- A weak non-resonant field so that the atom is always nearly in its ground state
- A perturbative solution to the Schrödinger equation

Recover, mutatis mutantis, all the results of the previous chapter with the harmonically bound electron model

Non-resonant interaction

Model

Incoming plane wave:

$$\mathbf{E}(0,t) = E_0 \mathbf{u}_z \cos\omega t \tag{37}$$

Hamiltonian

$$H = H_0 + H_1 \tag{38}$$

with

$$H_1 = -qZE_0 \cos\omega t \tag{39}$$

Interaction representation w.r.t. H_0

$$\widetilde{H} = U_0^\dagger H_1 U_0 \quad \text{with} \quad U_0 = \exp(-iH_0 t/\hbar) \tag{40}$$

Non-resonant interaction

Model

Expansion of the wave function over the eigenstates of H_0:

$$\left|\widetilde{\Psi}\right\rangle = \sum_j \beta_j \left|j\right\rangle \tag{41}$$

Injection in the Schrödinger equation and scalar product with $\langle k|$

$$i\hbar \frac{d\beta_k}{dt} = \sum_j \langle k| U_0^\dagger H_1 U_0 \left|j\right\rangle \beta_j \tag{42}$$

With $U_0 \left|j\right\rangle = \exp(-i\omega_j t) \left|j\right\rangle$, $\omega_j = E_j/\hbar$ and $\omega_{kj} = \omega_k - \omega_j$ (Bohr frequency)

$$\frac{d\beta_k}{dt} = -\frac{qE_0}{i\hbar} \sum_j e^{i\omega_{kj}t} \langle k| Z \left|j\right\rangle \beta_j \cos\omega t \tag{43}$$

Set of coupled first-order differential equations

Non-resonant interaction

Perturbative solution

Weak, non-resonant field. The atom is nearly in its ground state. Replace β_g by one (and all others by zero) in the r.h.s of the system

$$\frac{d\beta_k}{dt} \approx -\frac{qE_0}{i\hbar} e^{i\omega_{kg}t} \langle k| Z |g\rangle \cos\omega t \tag{44}$$

with the explicit solution

$$\beta_k(t) = \frac{qE_0}{2\hbar} \langle k| Z |g\rangle \left[\frac{e^{i(\omega_{kg}+\omega)t} - 1}{\omega_{kg}+\omega} + \frac{e^{i(\omega_{kg}-\omega)t} - 1}{\omega_{kg}-\omega} \right] \tag{45}$$

Resonances (and divergences) as expected at $\omega = \pm\omega_{kg}$ when $\langle k| Z |g\rangle$ does not vanish (selection rules). To compute the dipole, we return to the initial representation

$$|\Psi\rangle = \sum_k \beta_k e^{-i\omega_k t} |k\rangle \tag{46}$$

Non-resonant interaction

Comparison with classical model

Average dipole $\mathbf{D} = qZ\mathbf{u}_z = D\mathbf{u}_z$ (to be compared with the classical dipole)

$$\langle D \rangle = \sum_{\ell,k} \beta_\ell^* \beta_k e^{-i\omega_{k\ell}t} \langle \ell | \, qZ \, | k \rangle \tag{47}$$

Keeping only the first-order terms in the small β_k, $k \neq g$, amplitudes

$$\langle D \rangle = \sum_k \beta_k e^{-i\omega_{kg}t} \langle g | \, qZ \, | k \rangle + \text{c.c.} \tag{48}$$

$$\langle D \rangle = \frac{q^2 E_0}{2\hbar} \sum_k |\langle g | \, Z \, | k \rangle|^2 \left[\frac{e^{i\omega t} - e^{-i\omega_{kg}t}}{\omega_{kg} + \omega} + \frac{e^{-i\omega t} - e^{-i\omega_{kg}t}}{\omega_{kg} - \omega} + \text{c.c.} \right] \tag{49}$$

Non-resonant interaction

Comparison with classical model

Dipole contains terms oscillating permanently at the Bohr frequencies. They are an artifact of the model (transients damped in a more realistic model)

$$\langle D \rangle = \frac{q^2 E_0}{2\hbar} \sum_k |\langle g| Z |k\rangle|^2 \left[\frac{e^{i\omega t}}{\omega_{kg} + \omega} + \frac{e^{-i\omega t}}{\omega_{kg} - \omega} + \text{c.c.} \right] \tag{50}$$

Real quantum polarizability:

$$\langle D \rangle = \epsilon_0 \alpha_Q(\omega) E_0 \cos \omega t \tag{51}$$

$$\alpha_Q(\omega) = \frac{2q^2}{\hbar \epsilon_0} \sum_k |\langle g| Z |k\rangle|^2 \frac{\omega_{kg}}{\omega_{kg}^2 - \omega^2} \tag{52}$$

Non-resonant interaction

Comparison with classical model

Classical polarizability (ω_0: resonance frequency)

$$\alpha_c(\omega, \omega_0) = \frac{q^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2} \tag{53}$$

Hence

$$\alpha_Q(\omega) = \sum_k f_{kg} \alpha_c(\omega, \omega_{kg}) \tag{54}$$

with

$$f_{kg} = \frac{2m\omega_{kg}}{\hbar} |\langle g| Z |k\rangle|^2 \tag{55}$$

being the (real) oscillator strength

Non-resonant interaction

Oscillator strength sum rule

Rewrite

$$f_{kg} = \frac{2m\omega_{kg}}{\hbar} \langle g| Z |k\rangle \langle k| Z |g\rangle \tag{56}$$

Noting

$$[Z, H_0] = \frac{i\hbar}{m} P_z \ , \tag{57}$$

$$\langle k| P_z |g\rangle = \frac{m}{i\hbar} \langle k| ZH_0 - H_0 Z |g\rangle = -\frac{m\omega_{kg}}{i} \langle k| Z |g\rangle \tag{58}$$

Hence

$$f_{kg} = \frac{2}{i\hbar} \langle g| Z |k\rangle \langle k| P_z |g\rangle \tag{59}$$

Non-resonant interaction

Oscillator strength sum rule

Summing over k introduces a closure relation

$$\sum_k f_{kg} = \frac{2}{i\hbar} \langle g | ZP_z | g \rangle \tag{60}$$

f_{kg} being real, the r.h.s is equal to the half sum with its conjugate

$$\sum_k f_{kg} = \frac{1}{i\hbar} \langle g | ZP_z - P_z Z | g \rangle = 1 \tag{61}$$

A simple sum rule for the oscillator strengths

Non-resonant interaction

Comparison with classical model

In this picture, an atomic medium of numeric density N appears a a mixture of classical harmonically bound electrons with resonance frequencies ω_{kg} and densities Nf_{kg}. All our conclusions on the propagation of light in the classical medium thus retain their validity in this perturbative semi-classical model. This property explains why the naive harmonically bound electron leads to realistic predictions.

A two-level system

Consider now the case of a radiation resonant on the transition between between the two levels $|g\rangle$ (lower, possibly ground level) and $|e\rangle$ i.e.

$$\omega_0 = \omega_{eg}$$

All other levels can be neglected. Boils down to the interaction of a classical field with a spin 1/2 system.

Classical field and free atom

Atomic system

Two states $|e\rangle$ and $|g\rangle$ or $|+\rangle$ and $|-\rangle$ or $|0\rangle$ and $|1\rangle$ in quantum information science.

Operator basis set: Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{62}$$

$$[\sigma_x, \sigma_y] = 2i\sigma_z \tag{63}$$

Spin lowering and raising operators

$$\sigma_+ = |+\rangle\langle -| = \frac{\sigma_x + i\sigma_y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{64}$$

$$\sigma_- = |-\rangle\langle +| = \sigma_+^\dagger = \frac{\sigma_x - i\sigma_y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{65}$$

$$[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm \tag{66}$$

Classical field and free atom

Atomic system

Most general observable $\sigma_{\mathbf{u}}$ with $\mathbf{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\sigma_{\mathbf{u}} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \tag{67}$$

Eigenvectors

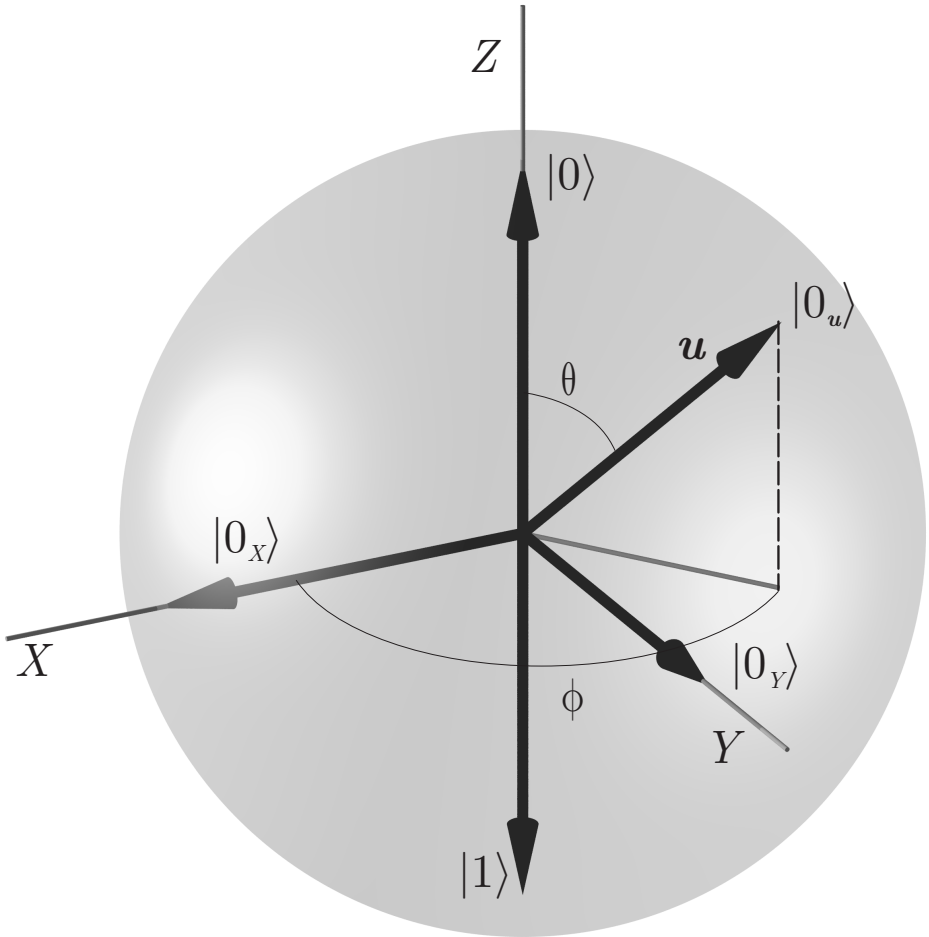
$$|+_{\mathbf{u}}\rangle = |0_{\mathbf{u}}\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle \tag{68}$$

$$|-_{\mathbf{u}}\rangle = |1_{\mathbf{u}}\rangle = -\sin\frac{\theta}{2}e^{-i\phi}|+\rangle + \cos\frac{\theta}{2}|-\rangle \tag{69}$$

Classical field and free atom

Atomic system

Bloch sphere



Classical field and free atom

Atomic system

Rotation on the Bloch sphere by an angle θ around the axis defined by $\mathbf{v}$

$$R_{\mathbf{v}}(\theta) = e^{-i(\theta/2)\sigma_{\mathbf{v}}} = \cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}\sigma_{\mathbf{v}} \tag{70}$$

e.g. angle θ around $\mathbf{u}_z$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{71}$$

with $R_z(\pi/2)\,|+_x\rangle = |+_y\rangle$ and $R_{\mathbf{v}}(2\pi) = -\mathbb{1}$

Classical field and free atom

Atomic Hamiltonian and observables

- Hamiltonian:

$$H_0 = \frac{\hbar\omega_{eg}}{2}\sigma_z \tag{72}$$

Generates a rotation of the Bloch vector at angular frequency ω_{eg} around Oz (Larmor precession in the NMR context).

- Dipole operator:

$$\mathbf{D} = \begin{pmatrix} 0 & \mathbf{d} \\ \mathbf{d}^* & 0 \end{pmatrix} = \mathbf{d}\sigma_x = \mathbf{d}(\sigma_+ + \sigma_-) \tag{73}$$

where $\mathbf{d}$ describes the polarization of the atomic transition. A priori complex, but taken as real for the sake of simplicity.

- Incoming field $\mathbf{E}(0,t) = \mathbf{E}_0 \cos(\omega t + \varphi)$. We note

$$E_1 = E_0 e^{-i\varphi} \tag{74}$$

Classical field and free atom

Atomic Hamiltonian and observables

Atom-field Hamiltonian:

$$H_1 = -\mathbf{d} \cdot \mathbf{E}_0 \cos(\omega t + \varphi)\sigma_x \tag{75}$$

$$H_1 = -\hbar\Omega \cos(\omega t + \varphi)\sigma_x \tag{76}$$

with definition of the 'Rabi frequency'

$$\Omega = \frac{\mathbf{d} \cdot \mathbf{E}_0}{\hbar} \tag{77}$$

Remove time dependence?

Classical field and free atom

Rabi precession

Introduce $H'_0 = \hbar\omega\sigma_z/2$ (inducing a spin precession at the field frequency) so that

$$H = H'_0 + \frac{\hbar\Delta}{2}\sigma_z + H_1 \tag{78}$$

with

$$\Delta = \omega_{eg} - \omega \ , \tag{79}$$

Interaction representation w.r.t. H'_0, defined by $U'_0 = \exp(-iH'_0 t/\hbar)$.

$$\widetilde{H} = U'^{\dagger}_0 H_1 U'_0 \tag{80}$$

σ_z part of H_1 unchanged (commutes with the evolution operator) but

$$\widetilde{\sigma}_{\pm} = U'^{\dagger}_0 \sigma_{\pm} U'_0 \tag{81}$$

Classical field and free atom

Rabi precession

Using the Baker-Hausdorff lemma:

$$e^{B} A e^{-B} = A + [B, A] + \frac{1}{2!} [B, [B, A]] + \dots \tag{82}$$

with $B \propto \sigma_z$ and $\sigma_+ = A$

$$\widetilde{\sigma}_+ = \sigma_+ + i\omega t \sigma_+ + (i\omega t)^2 \sigma_+ + \cdots = e^{i\omega t} \sigma_+ \tag{83}$$

and, by hermitic conjugation

$$\widetilde{\sigma}_- = e^{-i\omega t} \sigma_- \tag{84}$$

$$\widetilde{H} = \frac{\hbar\Delta}{2} \sigma_z - \frac{\hbar\Omega}{2} \left(e^{i(\omega t + \varphi)} + e^{-i(\omega t + \varphi)} \right) \left(e^{i\omega t} \sigma_+ + e^{-i\omega t} \sigma_- \right) \tag{85}$$

Two rapidly oscillating terms, and two constant ones.

Classical field and free atom

Rabi precession

Rotating wave approximation (RWA): neglect terms oscillating rapidly in $\widetilde{H}$

$$\widetilde{H} = \frac{\hbar\Delta}{2}\sigma_z - \frac{\hbar\Omega}{2}\left(\sigma_+ e^{-i\varphi} + \sigma_- e^{i\varphi}\right) = \frac{\hbar\Delta}{2}\sigma_z - \frac{\hbar\Omega}{2}\left(\sigma_x \cos\varphi + \sigma_y \sin\varphi\right) \tag{86}$$

$$H = \frac{\hbar\Omega'}{2}\sigma_{\mathbf{n}} \tag{87}$$

with

$$\mathbf{n} = \frac{\Delta\mathbf{u}_z - \Omega\cos\varphi\mathbf{u}_x - \Omega\sin\varphi\mathbf{u}_y}{\Omega'} \tag{88}$$

and

$$\Omega' = \sqrt{\Omega^2 + \Delta^2} \tag{89}$$

Hence,

$$U(t) = e^{-i(\Omega' t/2)\sigma_{\mathbf{n}}} = R_{\mathbf{n}}(\theta) \tag{90}$$

with

$$\theta = \Omega' t \tag{91}$$

Classical field and free atom

Rabi precession

Resonant case: rotation around an axis in the equatorial plane $\mathbf{n} = -\cos\varphi \mathbf{u}_x - \sin\varphi \mathbf{u}_y$. Choosing g as the initial state

$$p_e(t) = \frac{1 - \cos(\Omega t)}{2} \tag{92}$$

Rabi oscillation. Some particular pulses:

- '$\pi/2$ pulse', i.e. $t = \pi/2\Omega$. Evolution operator

$$R_{\mathbf{n}}(\pi/2) = \frac{1}{\sqrt{2}}(\mathbb{1} - i\sigma_{\mathbf{n}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & ie^{-i\varphi} \\ ie^{i\varphi} & 1 \end{pmatrix} \tag{93}$$

$$\begin{aligned} |g\rangle &\longrightarrow \frac{1}{\sqrt{2}} \left(|g\rangle + ie^{-i\varphi} |e\rangle \right) \\ |e\rangle &\longrightarrow \frac{1}{\sqrt{2}} \left(|e\rangle + ie^{i\varphi} |g\rangle \right) \end{aligned} \tag{94}$$

Classical field and free atom

Rabi precession

- $\Omega t = \pi$ (π-pulse) exchange of levels
- $\Omega t = 2\pi$ (2π pulse) global $-$ sign associated to a 2π rotation of a spin-1/2.

General case: rotation is around an axis making a non-trivial angle α (given by $\tan\alpha = \Omega'/\Delta$) with the downwards z axis. When starting from $|g\rangle$ the maximum excitation probability is

$$p_{e,m} = \frac{\Omega^2}{\Omega^2 + \Delta^2} \tag{95}$$

- Lorentzian resonance
- Width of order of π/τ for a given interrogation time τ

(no limit to the spectroscopic resolution since relaxation processes are not taken into account)

Classical field and free atom

Ramsey separated oscillatory fields method

Two short $\pi/2$ quasi-resonant pulses separated by a long time interval T. Assume $\varphi = -\pi/2$. The pulses induce the transformations:

$$|e\rangle \longrightarrow \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle) \tag{96}$$

$$|g\rangle \longrightarrow \frac{1}{\sqrt{2}}(-|e\rangle + |g\rangle) \tag{97}$$

Starting from $|g\rangle$, after pulse 1, atom is in state $|\Psi(\tau)\rangle = (1/\sqrt{2})(-|e\rangle + |g\rangle)$.

Classical field and free atom

Ramsey separated oscillatory fields method

During time T, the atom evolves under the Hamiltonian $(\hbar\Delta/2)\sigma_z$ and hence $|e\rangle \to \exp(-i\Phi/2)\,|e\rangle$ and $|g\rangle \to \exp(i\Phi/2)\,|g\rangle$, with $\Phi = \Delta t$.
State immediately before pulse 2, within an irrelevant global phase:

$$|\Psi(T)\rangle = \frac{1}{\sqrt{2}}\left(-|e\rangle + e^{i\Phi}\,|g\rangle\right) \tag{98}$$

Final state

$$|\Psi_f\rangle = -\frac{1}{2}\left[\left(1+e^{i\Phi}\right)|e\rangle + \left(1-e^{i\Phi}\right)|g\rangle\right) \tag{99}$$

$$p_e = \frac{1}{4}\left(1+e^{i\Phi}\right)^2 = \frac{1}{2}\left(1+\cos\Delta T\right) \tag{100}$$

(note that $p_e = 1$ for $\Delta = 0$: addition of two in-phase $\pi/2$ pulses).
Measurement of p_e provides a spectroscopic resolution of the order of $1/T$.

Classical field and free atom

Ramsey separated oscillatory fields method

Signal to noise discussion: N independent atoms undergoing the same Ramsey sequence

$$\langle N_e \rangle = \frac{N}{2}(1 + \cos\Phi) \tag{101}$$

with $\Phi = \Delta T$. Variance

$$\Delta^2 N_e = N p_e (1 - p_e) = \frac{N}{4}\sin^2\Phi \tag{102}$$

and hence

$$\Delta N_e = \frac{\sqrt{N}}{2}\sin\Phi \tag{103}$$

Two measurements for Δ and $\Delta + \delta$, with $\delta \ll 1/T$.

$$\langle N_e(\Delta + \delta)\rangle = \langle N_e(\Delta)\rangle - \frac{NT}{2}\delta \sin\Delta T \tag{104}$$

Classical field and free atom

Ramsey separated oscillatory fields method

Resolve the small detuning increment δ if

$$\frac{NT}{2}\delta \sin \Delta T > \sqrt{2}\frac{\sin \Delta T}{2}\sqrt{N} \tag{105}$$

or

$$\delta > \frac{\sqrt{2}}{T\sqrt{N}} \tag{106}$$

A more precise estimate of the spectroscopic sensitivity of the method. Amusingly independent of the interferometer phase. Ranges as $\sqrt{N}$ as expected for independent measurements.

Atomic relaxation

- Take into account spontaneous emission
- Take into account all other sources of damping
- Take into account fluctuating fields acting on the atom
- An opportunity to introduce the formal treatment of relaxation in quantum mechanics in a rather general frame: the Kraus operators and the Lindblad master equation

Atomic relaxation

System and environment

- Quantum system $\mathcal{S}$ (the atom here) coupled to an environment $\mathcal{E}$. Jointly in a pure state $|\Psi_{\mathcal{SE}}\rangle$.
- We are interested only in $\rho_{\mathcal{S}}$, obtained by tracing the projector $|\Psi_{\mathcal{SE}}\rangle\langle\Psi_{\mathcal{SE}}|$ over the environment (the state of the environment is forever inaccessible).
- We seek an evolution equation for $\rho_{\mathcal{S}}$ alone.

Atomic relaxation

Kraus operators

- Transformation of the system's density matrix during a short time interval

$$\rho(t) \longrightarrow \rho(t+\tau) \tag{107}$$

- $\tau \gg \tau_c$, correlation time of the reservoir observables, so that there are no coherent effects in the system-reservoir interaction
- This transformation is a 'quantum map'

$$\mathcal{L}(\rho(t)) = \rho(T+\tau) \tag{108}$$

Atomic relaxation

Kraus operators

Mathematical properties of a proper quantum map:

- Linear operation, i.e. a super-operator in a space of dimension $N_{\mathcal{S}}^2$ ($N_{\mathcal{S}}$ system's Hilbert space dimension).
- Preserve unit trace and positivity (a density operator does not have any negative eigenvalue).
- "Completely positive". If, at a time t, $\mathcal{S}$ entangled with $\mathcal{S}'$, $\mathcal{L}$ acting on $\mathcal{S}$ alone leads to a completely positive density operator for the joint state of $\mathcal{S}$ and $\mathcal{S}'$ (not all maps are completely positive e.g. partial transpose).

Atomic relaxation

Kraus operators

Any completely positive map can be written as

$$\mathcal{L}(\rho) = \sum_{\mu} M_{\mu} \rho M_{\mu}^{\dagger} \tag{109}$$

with the normalization condition

$$\sum_{\mu} M_{\mu}^{\dagger} M_{\mu} = \mathbb{1} \tag{110}$$

There are at most N_S^2 'Kraus' operators M_μ, which are not uniquely defined (same map when mixing the M_μ by a linear unitary matrix V: $M'\mu = V_{\mu\nu} M_\nu$).

Atomic relaxation

Kraus operators

Fit also in this representation:

- Hamiltonian evolution

$$\rho(t+\tau) = U(\tau)\rho U^\dagger(\tau) \tag{111}$$

- 'unread' generalized measurement

$$\rho \longrightarrow \sum_\mu O_\mu \rho O_\mu^\dagger \tag{112}$$

but not a measurement whose result μ is known

$$\rho \longrightarrow \frac{O_\mu \rho O_\mu^\dagger}{\mathrm{Tr}(O_\mu \rho O_\mu^\dagger)} \tag{113}$$

(non-linear normalization term in the denominator)

Atomic relaxation

Lindblad equation

Kraus representation and differential representation of the map

$$\rho(t+\tau) = \sum_{\mu} M_\mu \rho M_\mu^\dagger \approx \rho(t) + \frac{d\rho}{dt}\tau \tag{114}$$

- Environment unaffected by the system: the M_μs do not depend upon time t.
- They, however, depend clearly upon the tiny time interval τ.
- One and only one of the M_μs is thus of the order of unity and all others must then be of order $\sqrt{\tau}$.

$$M_0 = \mathbb{1} - iK\tau \tag{115}$$

$$M_\mu = \sqrt{\tau} L_\mu \qquad \text{for } \mu \neq 0 \tag{116}$$

Atomic relaxation

Lindblad equation

K, having no particular properties, can be split in hermitian and anti-hermitian parts:

$$K = \frac{H}{\hbar} - iJ \ , \tag{117}$$

where

$$H = \frac{\hbar}{2}\left(K + K^\dagger\right) \tag{118}$$

$$J = \frac{i}{2}\left(K - K^\dagger\right) \tag{119}$$

are both hermitian.

$$M_0 = \mathbb{1} - \frac{i\tau}{\hbar}H - J\tau \tag{120}$$

Atomic relaxation

Lindblad equation

Thus

$$M_0 \rho M_0^\dagger = \rho - \frac{i\tau}{\hbar}[H,\rho] - \tau[J,\rho]_+ \tag{121}$$

where $[J,\rho]_+ = J\rho + \rho J$ is an anti-commutator.

$$M_0^\dagger M_0 = \mathbb{1} - 2J\tau \quad \text{and thus, by normalization since} \quad \sum_\mu M_\mu^\dagger M_\mu = \mathbb{1} \tag{122}$$

$$J = \frac{1}{2}\sum_{\mu\neq 0} L_\mu^\dagger L_\mu \tag{123}$$

"Lindblad form" of the master equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_{\mu\neq 0}\left(L_\mu \rho L_\mu^\dagger - \frac{1}{2}L_\mu^\dagger L_\mu \rho - \frac{1}{2}\rho L_\mu^\dagger L_\mu\right) \tag{124}$$

Atomic relaxation

Quantum jumps

Consider a single time interval τ in the simple situation where the initial state is pure $\rho(0) = |\Psi\rangle\langle\Psi|$, with no Hamiltonian evolution. Then

$$\rho(\tau) = |\Psi\rangle\langle\Psi| + \tau \sum_{\mu} \left(L_\mu |\Psi\rangle\right)\left(\langle\Psi| L_\mu^\dagger\right) \tag{125}$$

- Density matrix at time τ is a statistical mixture of the initial pure state (with a large probability of order 1) and of projectors on the states $L_\mu |\Psi\rangle$.
- The L_μs are 'jump operators' which describe a random (possibly large) evolution of the system which suddenly (at the time scale of the evolution) changes under the influence of the environment.
- Intuitive picture of quantum jumps for an atom emitting a single photon

Atomic relaxation

Quantum jumps

- The quantum jump operators are not defined unambiguously. Again, the same master equation can be recovered from different sets of M_μs (or L_μs) linked together by a unitary transformation matrix. Different choices correspond to the so-called 'unravelings' of the master equation.
- In some situations, the quantum jumps have a direct physical meaning. e.g. emitting atom completely surrounded by a photo-detector array. The quantum jump then corresponds to a click of one detector. Different unravelings may then correspond to different ways of monitoring the environment, in this case to different detectors (photon counters, homodyne recievers...)
- In other situations, the quantum jumps are an abstract representation of the system+environment evolution.

Atomic relaxation

Quantum trajectories

- Even when the environment is not explicitly monitored, one may imagine that it is done. We then imagine we have full information about which quantum jump occurs when.
- The system is thus, at any time, in a pure state, which undergoes a stochastic trajectory in the Hilbert space, made up of continuous Hamiltonian evolutions interleaved with sudden quantum jumps.
- However, since we only imagine the information is available, we should describe the evolution of the density operator by averaging the system evolution over all possible trajectories.
- The 'environment simulator' concept provides a simple recipe to perform this averaging.

Atomic relaxation

Environment simulator

$\mathcal{B}$ coupled to $\mathcal{S}$ so that the reduced dynamics for $\mathcal{S}$ is the same as when coupled to $\mathcal{E}$.

- $\mathcal{B}$ prepared in the same reference state $|0\rangle$ at the start of each time interval τ
- Hamiltonian evolution of $\mathcal{S}+\mathcal{B}$ during the time interval τ

$$U_{\mathcal{SB}}|\Psi\rangle\otimes|0\rangle = M_0|\Psi\rangle\otimes|0\rangle + \sqrt{\tau}\sum_{\mu}(L_\mu|\Psi\rangle)\otimes|\mu\rangle \qquad (126)$$

- Unread measurement of $O_{\mathcal{B}}$ having the $|\mu\rangle$s as non degenerate eigenstates, with μ as the eigenvalue. This measurement tells which jump has happened if any.

Atomic relaxation

Environment simulator

At the end of the time interval τ:

- With a probability $p_0 = \langle\Psi| M_0^\dagger M_0 |\Psi\rangle = \mathrm{Tr}(\rho M_0^\dagger M_0) = 1 - \tau \sum_{\mu\neq 0} \mathrm{Tr}(\rho L_\mu^\dagger L_\mu) = 1 - \sum_{\mu\neq 0} p_\mu$, the result is 0, no jump and

$$\frac{M_0 |\Psi\rangle}{\sqrt{p_0}} = \frac{1 - iH\tau/\hbar - J\tau}{\sqrt{p_0}} |\Psi\rangle \tag{127}$$

  Evolution can be interpreted as resulting from evolution in the non-hermitian Hamiltonian

$$H_{eff} = H - i\hbar J \tag{128}$$

- With a probability $p_\mu = \tau \mathrm{Tr}(\rho L_\mu^\dagger L_\mu)$, the result is μ and the system's state is accordingly projected onto $M_\mu |\Psi\rangle / \sqrt{p_\mu} = L_\mu |\Psi\rangle / \sqrt{p_\mu/\tau}$.

The quantum trajectory is defined by the repetition of such steps.

Atomic relaxation

Environment simulator

We have no access to the environment state in most real cases.

- Recovers the right evolution during τ by averaging all projectors on all possible final pure states (with proper measurement probability weights).
- Recovers the full density operator evolution by averaging the projectors on all possible quantum trajectory states.
- Full mathematical equivalence between this average and the solution of the Lindblad equation.
- Leads to an efficient numerical method for solving Lindblad equations.

Atomic relaxation

Quantum Monte Carlo trajectories

- Initialize the state (randomly chosen eigenstate $|\Psi\rangle$ of ρ)
- For each time interval τ, evolve $|\Psi\rangle$ according to:
  - Compute $p_\mu = \tau \langle\Psi| L_\mu^\dagger L_\mu |\Psi\rangle$ and $p_0 = 1 - \sum_{\mu\neq 0} p_\mu$.
  - Use a (good) random number generator to decide upon the result of the measurement of $\mathcal{B}$.
  - If the result of the measurement is zero, evolve $|\Psi\rangle$ with

$$|\Psi\rangle \longrightarrow \frac{1 - iH\tau/\hbar - J\tau}{\sqrt{p_0}} |\Psi\rangle \tag{129}$$

  - If the result of the measurement is $\mu \neq 0$, evolve $|\Psi\rangle$ by:

$$|\Psi\rangle \longrightarrow \frac{L_\mu}{\sqrt{\langle\Psi| L_\mu^\dagger l_\mu |\Psi\rangle}} |\Psi\rangle = \frac{L_\mu}{\sqrt{p_\mu/\tau}} |\Psi\rangle \tag{130}$$

- Repeat the procedure for many trajectories
- Average the projectors $\rho(t) = \overline{|\Psi(t)\rangle \langle\Psi(t)|}$

Atomic relaxation

Quantum Monte Carlo trajectories

Interest of the Monte Carlo method:

- For each trajectory computes only a state vector with N_S dimensions i.e. N_S coupled differential equations, instead of N_S^2 equations for the full density operator.
- Neeeds a statistical sample of trajectories. A few hundreds is enough to get a qualitative solution. Method more efficient than the direct integration when N_S is larger than a few hundreds.
- Gives a physical picture of the relaxation process (see below).

An extremely useful method, with thousands of applications.

Atomic relaxation

Spontaneous emission

A practical (and important) example. Optical transition: Zero temperature model.
A single jump operator (describing photon emission in a downwards transition)

$$L = \sqrt{\Gamma}\sigma_- \tag{131}$$

with $\Gamma = 1/T_1$ ('longitudinal relaxation time'). Lindblad equation

$$\frac{d\rho}{dt} = \Gamma\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\sigma_+\sigma_-\rho - \frac{1}{2}\rho\sigma_+\sigma_-\right) \tag{132}$$

Atomic relaxation

Spontaneous emission

With

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} \tag{133}$$

the solution of the Lindblad equation is

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} \tag{134}$$

$$\frac{d\rho_{eg}}{dt} = -\frac{\Gamma}{2}\rho_{eg} \tag{135}$$

- Relaxation of excited state population with a rate Γ.
- Relaxation of coherence with a rate $\Gamma/2$ (compatible with $\rho_{eg} \leq \sqrt{\rho_{ee}\rho_{gg}}$)

Atomic relaxation

Phase damping

Model atomic relaxation due to random fields altering the atomic frequency and scrambling the coherence phase.

- Jump operator $\sqrt{\gamma/2}\sigma_z$ with $\gamma = 1/T_2$ the 'transverse' relaxation rate and T_2 the transverse relaxation time. Models sudden phase shifts of coherences.
- No damping of the populations, but coherences damped at rate γ.
- Complete Lindblad equation with spontaneous emission

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} \tag{136}$$

$$\frac{d\rho_{eg}}{dt} = -\frac{\Gamma}{2}\rho_{eg} - \gamma\rho_{eg} = -\gamma'\rho_{eg} \tag{137}$$

where we define the total relaxation rate of the coherence by:

$$\gamma' = \gamma + \frac{\Gamma}{2} \tag{138}$$

Atomic relaxation

Spontaneous emission

Case of an initial superposition state $|\Psi_0\rangle = (1/\sqrt{2})(|e\rangle + |g\rangle)$. Analysis in terms of the Monte Carlo trajectories.

- No jump evolution. With $|\Psi(t)\rangle = c_e |e\rangle + c_g |g\rangle$ and use effective Hamiltonian

$$H = -i\hbar J = -\frac{i\hbar}{2}\Gamma\sigma_+\sigma_- = -\frac{i\hbar}{2}\Gamma |e\rangle\langle e| \tag{139}$$

$$i\hbar\frac{dc_e}{dt} = -\frac{i\hbar}{2}\Gamma c_e \qquad c_e(t) = c_e(0)e^{-\Gamma t/2} \qquad \frac{dc_g}{dt} = 0 \tag{140}$$

$$|\Psi(t)\rangle = \frac{1}{|c_e(0)|^2 e^{-\Gamma T} + |c_g(0)|^2}\left(c_e(0)e^{-\Gamma t/2}|e\rangle + c_g(0)|g\rangle\right) \tag{141}$$

  A negative detection (no photon emitted) changes the system's state.

- Jump: state becomes $|g\rangle$. No further evolution.

Optical Bloch equations

- Merge the atom-field interaction and the relaxation (phase damping and/or spontaneous emission) in a single set of equations.
- Analyse the immediate consequences of these equations.

Optical Bloch equations

The equations

Hamiltonian in interaction representation w.r.t. the field frequency:

$$H = \frac{\hbar\Delta}{2}\sigma_z - \frac{\hbar\Omega}{2}\left(\sigma_+ e^{-i\varphi} + \sigma_- e^{i\varphi}\right) \tag{142}$$

with $\Omega = dE_0/\hbar$ and $\Delta = \omega_{eg} - \omega$ and

$$E_1 = E_0 e^{-i\varphi} \tag{143}$$

Optical Bloch equations

The equations

Coherent evolution of ρ ruled by the Schrödinger equation:

$$\begin{aligned} \frac{d\rho_{ee}}{dt} &= \Omega \mathrm{Im}\left(\rho_{eg} e^{i\varphi}\right) \\ &= \frac{d}{\hbar} \mathrm{Im}\left(\rho_{eg} E_1^*\right) \end{aligned} \tag{144}$$

and

$$\begin{aligned} \frac{d\rho_{eg}}{dt} &= -i\Delta\rho_{eg} + i\frac{\Omega}{2} e^{-i\varphi}(\rho_{gg} - \rho_{ee}) \\ &= -i\Delta\rho_{eg} - i\frac{d}{2\hbar} E_1(\rho_{ee} - \rho_{gg}) \end{aligned} \tag{145}$$

Optical Bloch equations

The equations

Add relaxation (assume mere addition of evolution terms and note that Lindblad equation terms are not changed in interaction representation)

$$\frac{d\rho_{ee}}{dt} = \frac{d}{\hbar}\mathrm{Im}\,(\rho_{eg}E_1^*) - \Gamma\rho_{ee} \tag{146}$$

$$\frac{d\rho_{eg}}{dt} = -i\Delta\rho_{eg} - i\frac{d}{2\hbar}E_1(\rho_{ee}-\rho_{gg}) - \gamma'\rho_{eg} \tag{147}$$

with $\Gamma = 1/T_1$ and $\gamma' = (1/2T_1) + 1/T_2$

Optical Bloch equations

Equivalent forms

Introducing

- The populations $N_e = \rho_{ee}$ and $N_g = \rho_{gg}$
- The complex dipole amplitude

$$\mathcal{D} = 2d\rho_{eg} \tag{148}$$

so that the average value of the dipole in state ρ is $\mathrm{Re}\,\mathcal{D}$

We get:

$$\frac{dN_e}{dt} = \frac{1}{2\hbar}\mathrm{Im}\,(\mathcal{D}E_1^*) - \Gamma N_e \tag{149}$$

$$\frac{d\mathcal{D}}{dt} = -i\Delta\mathcal{D} - \gamma'\mathcal{D} - i\frac{d^2E_1}{\hbar}(N_e - N_g) \tag{150}$$

Optical Bloch equations

Equivalent forms

Introducing the Bloch vector $\mathbf{r} = (x, y, z)$ so that

$$\rho = \frac{1 + \mathbf{r} \cdot \boldsymbol{\sigma}}{2} \tag{151}$$

or

$$\rho = \frac{1}{2} \begin{pmatrix} 1+z & x - iy \\ x + iy & 1 - z \end{pmatrix} \tag{152}$$

$$x = 2\mathrm{Re}\,\rho_{eg} \qquad y = -2\mathrm{Im}\,\rho_{eg} \qquad z = 2\rho_{ee} - 1 \tag{153}$$

With $E_1 = E_x + iE_y$

$$\frac{dz}{dt} = -\frac{d}{\hbar}(xE_y + yE_x) - \Gamma(1+z) \tag{154}$$

$$\frac{dx}{dt} = -\Delta y + \frac{d}{\hbar} z E_y - \gamma' x \tag{155}$$

$$\frac{dy}{dt} = +\Delta x + \frac{d}{\hbar} z E_x - \gamma' y \tag{156}$$

Optical Bloch equations

Rabi oscillations revisited

Rabi oscillations with relaxation. Simplifying hypotheses:

- Initial state $|g\rangle$ corresponding to $z = -1$ and $x = y = 0$.
- The field is purely real: $E_y = 0$, $E_x = +E_0$
- Atom and field are at resonance: $\Delta = 0$.

$$\frac{dz}{dt} = -\Omega y - \Gamma(1+z) \tag{157}$$

$$\frac{dy}{dt} = \Omega z - \gamma' y \tag{158}$$

$x = 0$ at any time.

$$\frac{d^2z}{dt^2} + (\Gamma + \gamma')\frac{dz}{dt} + \left(\Omega^2 + \gamma'\Gamma\right) z = -\gamma'\Gamma \tag{159}$$

Optical Bloch equations

Rabi oscillations revisited

Steady state:

$$z_s = -\frac{\gamma'\Gamma}{\Omega^2 + \gamma'\Gamma} \tag{160}$$

$$y_s = \frac{\Omega}{\gamma'} z = -\frac{\Omega\Gamma}{\Omega^2 + \gamma'\Gamma} \tag{161}$$

- For $\Omega \to 0$, $y_s = 0$ and $z_s = -1$
- For $\Omega \to \infty$, $z_s = y_s = 0$

Optical Bloch equations

Rabi oscillations revisited

Transient regime. Simplifying hypotheses:

- $\gamma' = \Gamma/2$: no transverse relaxation
- $\Omega \gg \Gamma$: Strong drive

$$\frac{d^2z}{dt^2} + \frac{3\Gamma}{2}\frac{dz}{dt} + \Omega^2 z = 0 \tag{162}$$

Solution:

$$z(t) = -\cos(\Omega t)e^{-3\Gamma t/2} \tag{163}$$

an exponentially damped Rabi oscillation at the frequency Ω.

Optical Bloch equations

Rabi oscillations revisited

A simple interpretation in terms of quantum trajectories (only spontaneous emission relaxation)

- Before the first jump, an uninterrupted Rabi oscillation
- The first jump projects the atom in $|g\rangle$ and restarts the Rabi oscillation
- The occurrence of random jumps thus dephase the oscillations corresponding to different trajectories
- Hence an exponential damping of the Rabi oscillation amplitude.

Optical Bloch equations

Oscillator strength revisited

Return to the hypotheses of first paragraph

- Atom initially in $|g\rangle$
- Detuned field $\Delta \gg \Gamma, \gamma'$, hence $N_g \approx 1$

We determine the steady state complex dipole $\mathcal{D} = 2d\rho_{ge}$ from

$$\frac{d\mathcal{D}}{dt} = -i\Delta\mathcal{D} - \gamma'\mathcal{D} - i\frac{d^2 E_1}{\hbar}(N_e - N_g) \tag{164}$$

$$\mathcal{D}_s = \frac{d^2}{\hbar\Delta}E_1 = \frac{q^2|\langle e|z|g\rangle|^2}{\hbar(\omega_{eg} - \omega)}E_1 \tag{165}$$

and define the quantum polarizability as

$$\alpha_Q = \frac{q^2}{\hbar(\omega_{eg} - \omega)}|\langle e|z|g\rangle|^2 \tag{166}$$

Optical Bloch equations

Oscillator strength revisited

Comparing the quantum and the classical polarizability:

$$\alpha_c = \frac{q^2}{2m\epsilon_0\omega_{eg}}\frac{1}{\omega_{eg}-\omega} \tag{167}$$

we get back the 'oscillator strength' (a mere consistency check)

$$f = \frac{2m\omega_{eg}}{\hbar}|\langle e|\, z\, |g\rangle|^2 \tag{168}$$

Two limit cases

Back to Einstein coefficients

Recover the Einstein coefficients as a limit case of the Optical Bloch Equations in two limit cases

- Strong transverse damping $\gamma' \approx \gamma$
- Stochastic, noisy driving field

In both cases, stochasticity turns the coherent Rabi oscillation into transfer rates à la Einstein

Two limit cases

Strong transverse relaxation

Assume $\gamma' \approx \gamma$ and $\Gamma \ll \gamma'$. Use again

$$\frac{d\mathcal{D}}{dt} = -i\Delta\mathcal{D} - \gamma'\mathcal{D} - i\frac{d^2 E_1}{\hbar}(N_e - N_g) \tag{169}$$

Fast relaxation allows to neglect $d\mathcal{D}/dt$. Assume thus that the dipole is at any time in the steady state value:

$$\mathcal{D} = \frac{i}{\gamma' + i\Delta}\frac{d^2 E_1}{\hbar}(N_g - N_e) \tag{170}$$

Inject in the equation of motion for N_e:

$$\begin{aligned}\frac{dN_e}{dt} &= -\Gamma N_e + \frac{1}{2\hbar}\mathrm{Im}\left[\frac{i}{\gamma' + i\Delta}\frac{d^2 E_1}{\hbar}(N_g - N_e)E_1^*\right] \\ &= -\Gamma N_e + \frac{d^2 E_0^2}{2\hbar^2}(N_g - N_e)\frac{\gamma'}{\gamma'^2 + \Delta^2}\end{aligned} \tag{171}$$

Two limit cases

Strong transverse relaxation

Assume a small but finite frequency bandwidth for the electric field: $E_0^2 \propto u_{\nu_0}$. Make field resonant ($\Delta = 0$).

$$\frac{dN_e}{dt} = -\Gamma N_e + \frac{d^2 E_0^2}{2\hbar^2 \gamma'}(N_g - N_e) = -\Gamma N_e + \frac{\Omega^2}{2\gamma'}(N_g - N_e) \tag{172}$$

$$\frac{dN_e}{dt} = A_{eg} N_e + (B_{ge} u_\nu N_g - B_{eg} u_\nu N_e) \tag{173}$$

with the evident correspondence $A_{eg} = \Gamma$

Two limit cases

Stochastic fields

Described in terms of a slowly variable complex amplitude $E_1(t)$ modulating an oscillation at the average frequency $\overline{\omega}$:

$$E(t) = E_1(t)e^{-i\overline{\omega}t} \tag{174}$$

Stochastic properties encoded in the autocorrelation function:

$$\Gamma_E(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} E_1^*(t')E_1(t'-\tau)\,dt' \tag{175}$$

or, within an ergodic hypothesis

$$\Gamma_E(\tau) = \overline{E_1^*(t)E_1(t-\tau)} \tag{176}$$

where the overline denotes an average over very many realizations of the source. Γ_E has a width τ_c (defining the source correlation time). Note

$$\Gamma_E(-\tau) = \overline{E_1^*(t)E_1(t+\tau)} = \overline{E_1^*(t'-\tau)E_1(t')} = \Gamma_E^*(\tau) \tag{177}$$

Two limit cases

Stochastic fields

Spectral density of radiation $S_E(\omega)$:

$$S_E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \, \Gamma_E(\tau) e^{-i\omega\tau} \tag{178}$$

real due to (177). Spectrum of the source: spectral density translated by $\overline{\omega}$. Width of the order of $1/\tau_c$.

Two limit cases

Stochastic fields

Since $E_1(t)$ varies slowly at the time scale of the optical frequency:

$$\frac{d\rho_{eg}}{dt} = -i\Delta\rho_{eg} - \gamma'\rho_{eg} - \frac{id}{2\hbar}E_1(t)(\rho_{ee} - \rho_{gg}) \tag{179}$$

where Δ is now $\omega_{eg} - \overline{\omega}$. Defining

$$\widetilde{\rho_{eg}} = \rho_{eg}e^{(i\Delta+\gamma')t} \tag{180}$$

we get

$$\widetilde{\rho_{eg}}(t) = -\frac{id}{2\hbar}\int_0^t E_1(t')(\rho_{ee} - \rho_{gg})(t')e^{(i\Delta+\gamma')t'}\,dt' \tag{181}$$

With $\rho_{eg} = \widetilde{\rho_{eg}}\exp[-(i\Delta + \gamma')t]$:

$$\rho_{eg}(t) = -\frac{id}{2\hbar}\int_0^t E_1(t')(\rho_{ee} - \rho_{gg})(t')e^{(-i\Delta-\gamma')(t-t')}\,dt' \tag{182}$$

Two limit cases

Stochastic fields

Plug the expression of $\rho_{eg}(t)$ in the equation of the populations:

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} - \frac{d^2}{2\hbar^2}\mathrm{Re}\int_0^t E_1^*(t)E_1(t')(\rho_{ee}-\rho_{gg})(t')e^{(-i\Delta-\gamma')(t-t')}\,dt' \tag{183}$$

Settting $t - t' = \tau$, or $t' = t - \tau$ $(0 \leq \tau \leq t)$

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} - \frac{d^2}{2\hbar^2}\mathrm{Re}\int_0^t E_1(t-\tau)E_1^*(t)(\rho_{ee}-\rho_{gg})(t-\tau)e^{(-i\Delta-\gamma')\tau}\,d\tau \tag{184}$$

Two limit cases

Stochastic fields

Perform an ensemble average of the evolution equations (leaving ρ invariant)

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} - \frac{d^2}{2\hbar^2}\mathrm{Re}\int_0^t \Gamma_E(\tau)(\rho_{ee}-\rho_{gg})(t-\tau)e^{(-i\Delta-\gamma')\tau}\,d\tau \quad (185)$$

Short source correlation time τ_c.

- Replace $(\rho_{ee}-\rho_{gg})(t-\tau)$ by $(\rho_{ee}-\rho_{gg})(t)$
- Extend upper integral bound to infinity

Two limit cases

Stochastic fields

Final equation of motion:

$$\frac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee} - C(\Delta)(\rho_{ee} - \rho_{gg}) \tag{186}$$

where

$$C(\Delta) = \frac{d^2}{2\hbar^2}\mathrm{Re}\int_0^\infty \Gamma_E(\tau)e^{(-i\Delta-\gamma')\tau}\,d\tau \tag{187}$$

Neglect the transverse relaxation rate γ' compared to the field frequency width.

$$C(\Delta) = \frac{d^2}{2\hbar^2}\mathrm{Re}\int_0^\infty \Gamma_E(\tau)e^{-i\Delta\tau}\,d\tau \tag{188}$$

Two limit cases

Stochastic fields

Link with spectral density:

$$2\pi S_E(\Delta) = \int_{-\infty}^{0} \Gamma_E(\tau) e^{-i\Delta\tau}\, d\tau + \int_{0}^{\infty} \Gamma_E(\tau) e^{-i\Delta\tau}\, d\tau \tag{189}$$

With $\Gamma_E(-\tau) = \Gamma_E^*(\tau)$:

$$\int_{-\infty}^{0} \Gamma_E(\tau) e^{-i\Delta\tau}\, d\tau = \int_{0}^{\infty} \Gamma_E(-\tau) e^{i\Delta\tau}\, d\tau = \left(\int_{0}^{\infty} \Gamma_E(\tau) e^{-i\Delta\tau}\, d\tau \right)^* \tag{190}$$

Hence,

$$2\pi S_E(\Delta) = 2\mathrm{Re} \int_{0}^{\infty} \Gamma_E(\tau) e^{-i\Delta\tau}\, d\tau \tag{191}$$

and, finally

$$C(\Delta) = \frac{\pi d^2}{2\hbar^2} S_E(\Delta) \tag{192}$$

Two limit cases

Einstein at last

Assuming finally the resonance condition ($\Delta = 0$) and noting that

$$u_\nu = 2\pi^2 \epsilon_0 S_E(0) \tag{193}$$

we get

$$C(0) = C = B_{eg} u_\nu \tag{194}$$

and

$$B_{eg} = \frac{d^2}{4\pi\epsilon_0 \hbar^2} \tag{195}$$

and finally

$$\frac{dN_e}{dt} = -A_{eg} N_e + B_{eg} u_\nu (N_g - N_e) \tag{196}$$

Two limit cases

Einstein at last

The value of B_{eg} obtained here differs by a factor $3/2$ from that obtained from

$$A_{eg} = \frac{d^2\omega^3}{3\pi\epsilon_0\hbar c^3} \tag{197}$$

which is

$$B_{eg} = \frac{d^2}{6\pi\epsilon_0\hbar^2} \tag{198}$$

Reason: no averaging over polarizations in our calculation.

Two limit cases

Spectrum of a lamp

An exercise on autocorrelation functions. Spontaneous emission by a large ensemble of atoms. Train of exponentially damped pulses (N_p per unit time) with random relative phases:

$$E_1(t) = \sum_{i=-\infty}^{\infty} E_0 e^{i\phi_i} e^{-(t-t_i)/\tau_e} \Theta(t-t_i) \tag{199}$$

$$\Gamma_E = N_p T \gamma_e \tag{200}$$

with

$$\gamma_E(\tau) = \frac{1}{T} E_0^2 \int_0^T e^{-t/\tau_e} e^{-(t-\tau)/\tau_e} \Theta(t-\tau)\, dt \tag{201}$$

$$\begin{aligned} \gamma_E(\tau) &= \frac{1}{T} E_0^2 \left[\int_\tau^\infty e^{-2t/\tau_e}\, dt \right] e^{\tau/\tau_e} \\ &= \frac{1}{T} E_0^2 \frac{\tau_e}{2} e^{-|\tau|/\tau_e} \end{aligned} \tag{202}$$

Two limit cases

Spectrum of a lamp

Finally

$$\Gamma_E(\tau) = N_p E_0^2 \frac{\tau_e}{2} e^{-|\tau|/\tau_e} \tag{203}$$

and

$$S_E(\omega) = \frac{N_p E_0^2}{\pi} \frac{1}{\omega^2 + (1/\tau_e)^2} \tag{204}$$

a Lorentzian spectrum with a width $1/\tau_e$.

Applications

Explore direct applications of the Optical Bloch equations:

- Steady-state and Saturation
- Optical pumping
- Dark resonance and EIT
- Light shifts and Autler Townes splitting
- Maxwell Bloch equations

Steady-state and Saturation

Classical model (chapter 1): power given to the matter by the field

$$\mathcal{E} = \frac{1}{2}\epsilon_0 \omega \chi'' |E_1|^2 = \frac{1}{2}\epsilon_0 \omega \mathcal{N} \alpha'' |E_1|^2 \tag{205}$$

where $\mathcal{N}$ is the number of atoms in the medium (the populations in the Bloch equations sum to one so that the number of atoms in $|e\rangle$ is $\mathcal{N} N_z$) The complex dipole amplitude D is $D = \epsilon_0 \alpha E_1$ and thus

$$\mathcal{E} = \mathcal{N} \frac{\omega E_1}{2} \mathrm{Im}\, D \tag{206}$$

Linear function of the incoming power. An unrealistic model: an atom cannot diffuse a MW laser field. What is the prediction of the OBEs?

Steady-state and Saturation

Steady state power

Replace in the classical expression of the energy exchange the dipole by $\mathcal{D}$. Recall the OBEs and assume E_1 real without loss of generality

$$\frac{dN_e}{dt} = \frac{1}{2\hbar}\mathrm{Im}\left(\mathcal{D}E_1\right) - \Gamma N_e \tag{207}$$

$$\frac{d\mathcal{D}}{dt} = -i\Delta\mathcal{D} - \gamma'\mathcal{D} - i\frac{d^2E_1}{\hbar}(N_e - N_g) \tag{208}$$

In the steady state:

$$\mathcal{D} = \frac{\Delta + i\gamma'}{\Delta^2 + \gamma'^2}\frac{d^2E_1}{\hbar}(N_g - N_e) \tag{209}$$

Steady-state and Saturation

Steady state power

Similarly, the steady state value of N_e is

$$N_e = \frac{d^2 E_1^2}{2\hbar^2 \Gamma}(N_g - N_e)\frac{\gamma'}{\Delta^2 + \gamma'^2} \tag{210}$$

Introducing the Rabi frequency $\Omega = dE_1/\hbar$ and defining the saturation parameter:

$$s = \frac{\Omega^2}{\Gamma\gamma'}\frac{1}{1 + \Delta^2/\gamma'^2} \ , \tag{211}$$

which has a Lorentzian variation with the atom-field detuning Δ, we arrive at

$$N_e = \frac{s/2}{1+s} \tag{212}$$

$$N_g - N_e = \frac{1}{1+s} \ , \tag{213}$$

Steady-state and Saturation

Steady state power

We get also $\mathcal{D}$ such that

$$|\mathcal{D}|^2 = d^2 \frac{\Gamma}{\gamma'} \frac{s}{(1+s)^2} \ . \tag{214}$$

and finally

$$\mathcal{E} = \frac{\mathcal{N}\hbar\omega\Gamma}{2} \frac{s}{1+s} \ , \tag{215}$$

always positive, since there can be no population inversion. The absorbed energy has a Lorentzian shape for a small saturation parameter ($s \ll 1$; small Rabi frequency).

Steady-state and Saturation

Saturation intensity

At resonance ($\Delta = 0$) the 'saturation parameter' $s = s_0$ is:

$$s_0 = \frac{\Omega^2}{\Gamma\gamma'} \tag{216}$$

and

$$\mathcal{E} = \mathcal{N}\frac{\hbar\omega}{2}\Gamma\frac{s_0}{1+s_0} \tag{217}$$

- At low power, $\mathcal{E}$ is proportional to s_0 i.e. to the incoming field intensity. Recover classical model result
- At infinite input power,

$$\mathcal{E}_s = \mathcal{N}\hbar\omega\frac{\Gamma}{2} \tag{218}$$

  photons scattered at a rate $\Gamma/2$.
- Onset of the saturation for $s_0 \approx 1$

Steady-state and Saturation

Saturation intensity

With $s_0 = d^2 E_1^2/\hbar^2\Gamma\gamma'$ and an incident power per unit surface $I = \epsilon_0 c E_1^2/2$ then

$$s_0 = \frac{d^2 E_1^2}{\hbar^2\Gamma\gamma'} = \frac{I}{I_s} \tag{219}$$

where the saturation intensity I_s is

$$I_s = \frac{\Gamma\gamma'}{d^2}\frac{\epsilon_0 c}{2}\hbar^2 \tag{220}$$

Consider the simple case $\gamma' = \Gamma/2$ (no additional transverse damping) then

$$I_s = \frac{\Gamma^2}{4}\frac{\epsilon_0 c}{d^2}\hbar^2 \tag{221}$$

Steady-state and Saturation

Saturation intensity

Using (anticipating again on Chapter 4)

$$\Gamma = \frac{\omega^3 d^2}{3\pi\epsilon_0 \hbar c^3} \tag{222}$$

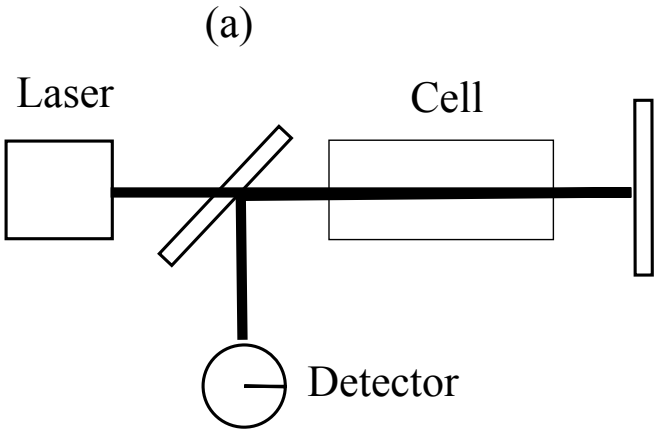
$$I_s = \frac{\pi}{3}\hbar\omega\Gamma\frac{1}{\lambda^2} = \hbar\omega\frac{\Gamma}{2}\frac{1}{\sigma_c} \tag{223}$$

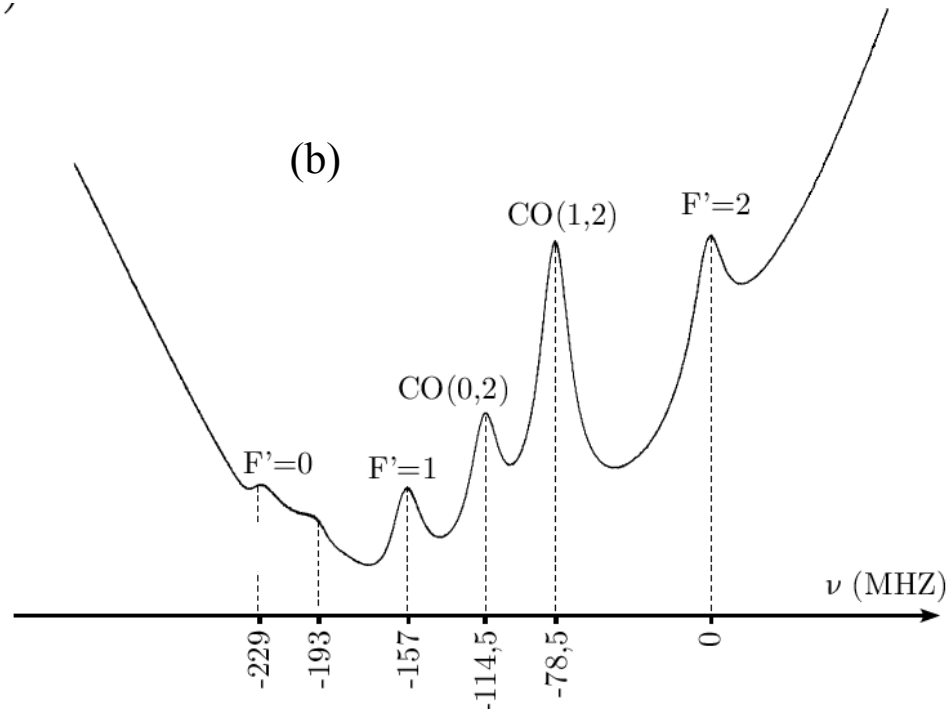
- Saturation: one photon incident in the resonant cross section of the classical model, $\sigma_c = 3\lambda^2/2\pi$, at the maximum rate of diffusion $\Gamma/2$.
- Order of magnitude: with $\Gamma = 3.10^7$ s^{-1}, $\lambda = 1$ μm we get $I_s = 0.6$ mW/cm^2

Steady-state and Saturation

Saturation spectroscopy

A useful method to get rid of the Doppler broadening of atomic transitions.





Steady-state and Saturation

Saturation spectroscopy

Resonance conditions for the two beams

- Direct beam: $\Delta = \omega_{eg} - \omega = -kv_z$
- Reflected beam: $\Delta = kv'_z$
  - Out of resonance (Δ much larger than Ω and Γ), the two counterpropagating beams interact with different velocity classes due to the Doppler effect. The absorptions are independent and equivalent to one path in a medium with a double atom number $2\mathcal{N}$ [1]. The absorbed energy is

$$\mathcal{E} = 2 \times \frac{\mathcal{N}\hbar\Omega\Gamma}{2} \frac{s_0}{1+s_0} \tag{224}$$

  - At resonance ($\Delta = 0$), the two beams interact with the $v_z = 0$ class. The saturation parameter is doubled (twice the intensity) but the atom number is twice lower (only one class). The absorbed energy is them

$$\mathcal{E}_0 = \frac{\mathcal{N}\hbar\Omega\Gamma}{2} \frac{2s_0}{1+2s_0} \tag{225}$$

[1] This quantity should be defined more carefully in terms of the Maxwell velocity distribution and the width of the velocity class.

Steady-state and Saturation

Saturation spectroscopy

Hence, the 'dip depth' is

$$\frac{\mathcal{E}_0}{\mathcal{E}} = \frac{1+s_0}{1+2s_0} \tag{226}$$

and its width is $\gamma'\sqrt{1+s_0}$. The best compromise corresponds to $s_0 \approx 1$, with a depth of $1/3$ and a width of $2\gamma'$).

Steady-state and Saturation

Saturation spectroscopy

Case of a multilevel atom: two nearly degenerate ground states, $|g\rangle$ and $|f\rangle$, and an excited state $|e\rangle$ (Λ system). Saturation resonances at:

- ω_{ge}
- ω_{fe}
- Crossover resonance dip: direct beam resonant at ω_{fe} for $kv_z = \omega - \omega_{fe}$, saturating the $f \to e$ transition, and reflected beam probing this saturation when resonant on $|g\rangle \to |e\rangle$ if $kv_z = -(\omega - \omega_{ge})$ i.e.

$$\omega = \frac{\omega_{fe} + \omega_{ge}}{2} \tag{227}$$

Optical pumping

Principle

Λ system again. $|e\rangle$ decays towards both $|g\rangle$ and $|f\rangle$ with rates Γ_{eg} and Γ_{ef}. Goal: populate only $|f\rangle$.
Method: drive selectively $|g\rangle \rightarrow |e\rangle$.
After a few fluorescence cycles, $|g\rangle$ is depopulated.
More quantitative approach based on Einstein's coefficients.

$$\frac{dN_e}{dt} = -(\Gamma_{eg} + \Gamma_{ef})N_e + \frac{\Omega^2}{2\gamma'}(N_g - N_e) \quad (228)$$

$$\frac{dN_g}{dt} = \Gamma_{eg} N_e - \frac{\Omega^2}{2\gamma'}(N_g - N_e) \quad (229)$$

$$\frac{dN_f}{dt} = \Gamma_{ef} N_e \quad (230)$$

with $N = N_e + N_f + N_g$

Optical pumping

Dynamics

- Steady state: $N_e = 0$ and hence $N_g = 0$ and $N_f = N$.
- Puming dynamics. In the weak pump limit:

$$\frac{\Omega^2}{\gamma'} \ll \Gamma_{eg},\ \Gamma_{ef} \tag{231}$$

N_e is low and at any time at a steady state value

$$N_e = \frac{\Omega^2/2\gamma'}{\Gamma_{eg} + \Gamma_{ef} + \Omega^2/2\gamma'} N_g \approx \frac{\Omega^2/2\gamma'}{\Gamma_{eg} + \Gamma_{ef}} N_g \tag{232}$$

$$\frac{dN_g}{dt} = -\Gamma_p N_g \tag{233}$$

where we define the optical pumping rate Γ_p by:

$$\Gamma_p = \frac{\Gamma_{ef}}{\Gamma_{eg} + \Gamma_{ef}} \frac{\Omega^2}{2\gamma'} \tag{234}$$

An exponential approach to the steady state.

Dark resonances and EIT

Dark states

Again Λ system with two resonant laser fields $\mathbf{E}_1$ and $\mathbf{E}_2$ separately coupled to the $|g\rangle \to |e\rangle$ and $|f\rangle \to |e\rangle$ transitions (selection rules make $|g\rangle$ impervious to E_2 and $|f\rangle$ to E_1. The total interaction Hamiltonian is $H_i = -\mathbf{d}\cdot(\mathbf{E}_1 + \mathbf{E}_2)$, where $\mathbf{d}$ is the dipole operator (back to the basics). We set

$$d_{eg} = \langle e|\mathbf{d}\cdot\mathbf{E}_1|g\rangle; \qquad d_{ef} = \langle e|\mathbf{d}\cdot\mathbf{E}_2|f\rangle \tag{235}$$

A state $|\Psi_-\rangle = c_g\,|g\rangle + c_f\,|f\rangle$ is decoupled from the lasers ($\langle e|H_i\,|\Psi_-\rangle = 0$) if $c_g d_{eg} + c_f d_{ef} = 0$ i.e.

$$c_g = \frac{d_{ef}}{d} \qquad \text{and} \qquad c_f = -\frac{d_{eg}}{d} \tag{236}$$

or

$$|\Psi_-\rangle = \frac{d_{ef}}{d}\,|g\rangle - \frac{d_{eg}}{d}\,|f\rangle \tag{237}$$

with

$$d = \sqrt{|d_{eg}|^2 + |d_{ef}|^2} \tag{238}$$

Dark resonances and EIT

Dark states

We have here written the states and the fields at a given time. This condition remains valid at all times if $|g\rangle$ and $|f\rangle$ are degenerate with lasers at resonance.

When $|g\rangle$ and $|f\rangle$ have different energies the dark state condition is maintained at any time if

$$|\Psi_-\rangle(t) = \frac{d_{ef}}{d} e^{-i\omega_1 t} e^{-i\omega_g t} |g\rangle - \frac{d_{eg}}{d} e^{-i\omega_2 t} e^{-i\omega_f t} |f\rangle \tag{239}$$

is, within a global phase, independent of time i.e; if $\omega_1 + \omega_g = \omega_2 + \omega_f$ or

$$\omega_1 - \omega_2 = \omega_{fg} \ , \tag{240}$$

if the difference of the fields frequencies is equal to the Bohr frequency between the two ground states. This is nothing but a Raman resonance condition.

The first evidence of dark resonances has been obtained in 1976 by Gozzini and his group in Pisa

Dark resonances and EIT

Dark states

The orthogonal state $|\Psi_+\rangle$ is maximally coupled to lasers:

$$|\Psi_+\rangle = \frac{d_{eg}^*}{d}|g\rangle + \frac{d_{ef}^*}{d}|f\rangle \tag{241}$$

In presence of relaxation (spontaneous emission), we have one ground state coupled to the laser and another uncoupled. This is again an optical pumping situation. After a few emission cycles, we unconditionnaly end up in the dark state $|\Psi_-\rangle$. Fluorescence stops.

Dark resonances and EIT

Electromagnetically induced transparency

Λ system, strong drive of $|f\rangle \to |e\rangle$ (E at frequency ω, Rabi frequency Ω), weak probe of $|g\rangle \to |e\rangle$ (E', ω', Ω'). Generalize OBES as:

$$\frac{d\rho_{ee}}{dt} = -(\Gamma_{ef} + \Gamma_{eg})\rho_{ee} - \Omega \mathrm{Im}\,\rho_{ef} - \Omega' \mathrm{Im}\,\rho_{eg} \quad (242)$$

$$\frac{d\rho_{ff}}{dt} = \Gamma_{ef}\rho_{ee} + \Omega \mathrm{Im}\,\rho_{ef} \quad (243)$$

$$\frac{d\rho_{gg}}{dt} = \Gamma_{eg}\rho_{ee} + \Omega' \mathrm{Im}\,\rho_{eg} \quad (244)$$

$$\frac{d\rho_{ef}}{dt} = -\frac{\Gamma_{ef}}{2}\rho_{ef} + i\frac{\Omega}{2}(\rho_{ee} - \rho_{ff}) + i\frac{\Omega'}{2}\rho_{gf} \quad (245)$$

$$\frac{d\rho_{eg}}{dt} = -\left(\frac{\Gamma_{eg}}{2} + i\Delta'\right)\rho_{eg} + i\frac{\Omega'}{2}(\rho_{gg} - \rho_{ee}) + i\frac{\Omega}{2}\rho_{fg} \quad (246)$$

$$\frac{d\rho_{gf}}{dt} = -i\Delta'\rho_{gf} + i\frac{\Omega}{2}\rho_{ge} - i\frac{\Omega'}{2}\rho_{ef} \quad (247)$$

Dark resonances and EIT

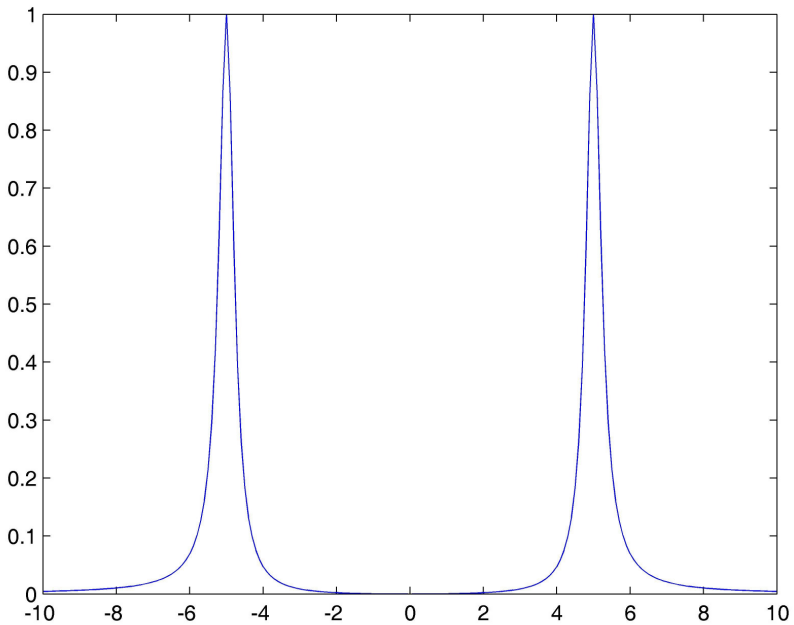
Electromagnetically induced transparency

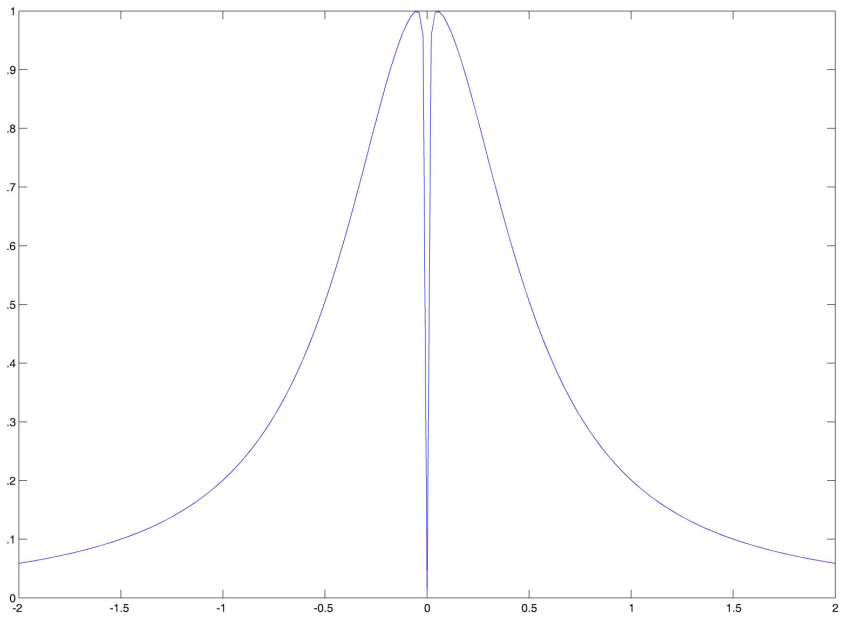
Simplify with $\Gamma_{ef} = \Gamma_{eg} = \Gamma$ and extract absorption of weak probe (absorbed energy $\mathcal{E}'$, proportional to $\mathrm{Im}\,\rho_{eg}$ in the steady state).

$$\mathcal{E}' \propto \frac{\Gamma^2 \Delta'^2}{\Gamma^2 \Delta'^2 + 4\left(\Delta'^2 - \Omega^2/4\right)^2} \tag{248}$$

Dark resonances and EIT

Electromagnetically induced transparency





Absorption of the probe field as a function of the detuning Δ for $\Omega = 10\Gamma$ (left) or $\Omega = 0.1\Gamma$ (right). The unit of the horizontal axis is Γ.

Dark resonances and EIT

Electromagnetically induced transparency

Slow light. In the small Ω case, the absorption has a narrow dip at resonance, whose width is only limited by the Rabi frequency. In actual experiments, the width can be only a few kHz. Hence, the (at resonance) transparent medium has an index of refraction n for the weak probe field varying rapidly with the detuning Δ i.e. with ω'. Since

$$v_g = \frac{c}{n + \omega' \, dn/d\omega'} \ , \tag{249}$$

the light group velocity can be made very small, 17 m/s in the original paper by Lene Hau and her group (Nature, 397, 594).

Maxwell Bloch equations

Treat propagation in an atomic medium. Maxwell:

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{250}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0 \tag{251}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{252}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \frac{\partial \mathbf{P}}{\partial t} \tag{253}$$

Hence, for a transverse wave

$$\Delta \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} \tag{254}$$

(note that we assume a low density and treat the macroscopic fields as being the local ones)

Maxwell-Bloch equations

Monochromatic plane wave in an isotropic medium

$$\mathbf{P} = P_0(z,t)e^{i(kz-\omega t)}\mathbf{u}_x \qquad \text{and} \qquad \mathbf{E} = E_0(z,t)e^{i(kz-\omega t)}\mathbf{u}_x \tag{255}$$

Hence, noting

$$\frac{\partial P_0}{\partial t} \ll \omega P_0 \tag{256}$$

and neglecting the proper time derivatives

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = i\frac{\mu_0\omega^2}{2k}P_0 \tag{257}$$

with

$$\mathbf{P}_0 = N\mathcal{D} = 2Nd\rho_{eg} \tag{258}$$

$$\frac{\partial E_0}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = i\frac{\omega Nd}{\epsilon_0 c}\rho_{eg} \tag{259}$$

Maxwell-Bloch equations

Pulse propagation

A simple application: propagation in a relaxation-free medium. Atoms described by the angle $\phi(z,t)$ of the Bloch vector with the vertical axis. Assuming real quantities

$$\frac{d\phi(z,t)}{dt} = \frac{dE_0(z,t)}{\hbar} = \Omega(z,t) \tag{260}$$

$$\rho_{eg} = -i \sin\frac{\phi}{2} \tag{261}$$

Hence

$$\frac{\partial^2\phi}{\partial z \partial t} + \frac{1}{c}\frac{\partial^2\phi}{\partial t^2} = -\mu \sin\phi \tag{262}$$

where

$$\mu = \frac{\omega N d^2}{2\epsilon_0 \hbar c} \tag{263}$$

Maxwell-Bloch equations

Pulse propagation

Using as independent variables z and the 'retarded time' $\tau = t - z/c$:

$$\frac{\partial^2 \phi}{\partial z \partial \tau} - \mu \sin \phi \tag{264}$$

Sine-Gordon equation