Master ICFP — Quantum Physics

Atoms and photons

Exam of December 19, 2013

Jean-Michel Raimond, Julien Laurat, Hélène Perrin

Duration three hours. Lectures notes (even on tablets), exercise class notes and manuscript documents are authorized. Books, mobile phones, computers, optical pumping are not.

Preliminary

This problem deals with the intermittent fluorescence observed for the first time on trapped ions by the Toschek, Dehmelt and Wineland groups simultaneously in 1986. We will take here the example of the Wineland experiment (PRL **57**, 1699) on mercury ions. A single mercury ion is held nearly at rest in an appropriate configuration of dc and ac electric fields (Paul trap). Figure 1(a) presents a simplified level scheme. The ground state, simply noted $|g\rangle$ in the following, is linked by two dipole transitions to two excited levels, $|e\rangle$ and $|f\rangle$ (levels 5D6P and 5D6S respectively, considered here as non-degenerate levels). We will call the $|g\rangle \rightarrow |e\rangle$ transition the "blue" transition, with angular frequency ω_e and wavelength $\lambda_e = 194$ nm. The $|g\rangle \rightarrow |f\rangle$ transition, with a longer wavelength $\lambda_f = 281$ nm (angular frequency ω_f), is called the "red" transition. Both excited levels spontaneously decay towards $|g\rangle$. The respective spontaneous emission rates are noted Γ_e and Γ_f. The lifetime, τ_e, of the $|e\rangle$ level is 2.3 ns, that of $|f\rangle$, τ_f, is much longer, of the order of 0.1 s.

A strong resonant laser drives the $|g\rangle \rightarrow |e\rangle$ transition above saturation. The ion, initially in $|g\rangle$, emits photons collected by appropriate optics and focused on a photomultiplier. A much weaker laser drives the $|g\rangle \rightarrow |f\rangle$ transition. The fluorescence observed on the blue line is then intermittent [see the actual signal in figure 1(b)]. This problem aims at explaining and quantitatively interpreting this intermittence using the waiting time statistics introduced by C. Cohen-Tannoudji and J. Dalibard (EPL, **1**, 441).

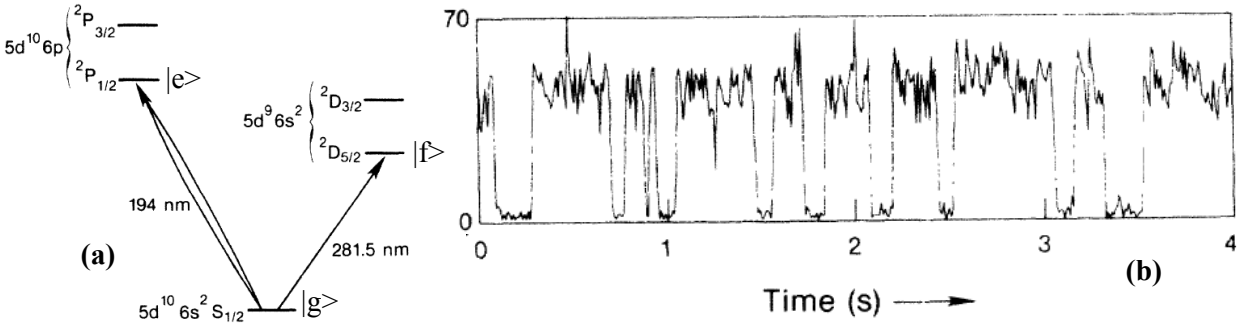

Figure 1: (a) Simplified level scheme of the Mercury ion. (b) Typical quantum jump signal. The time in in second units, and the vertical axis is the number of photons detected in a 10 ms time interval

I. Introduction

1. Give a qualitative explanation of the fluorescence intermittence in terms of quantum jumps. What is, from Figure 1(b), the estimated average duration of the "dark' and "bright" periods.

Is it possible to predict these values by intuitive arguments?

2. Give the numerical values of ω_e, ω_f, Γ_e, Γ_f. How many photons does the ion scatter per second during the bright periods? Compare that to the observed signal levels. Comment the difference. Do you think you could see the ion with your naked eye (if the bright blue transition was in the visible, of course)?

II. Two-level system

The quantum jumps are not easily predicted from the Optical Bloch equations (see Javanainen, PRA **33**, 2121). We will use instead the waiting time statistics. We first discuss them in the simple context of a two-level system before switching to the three-level situation of the mercury ion.

3. We consider thus a two-level atom ($|g\rangle \to |e\rangle$ transition, frequency ω_e, spontaneous emission rate Γ_e, zero temperature for the environment). We assume that the atom is initially in $|e\rangle$. What is(are) the quantum jump operator(s)? What is the non-hermitian Hamiltonian, H_e, describing the evolution between jumps?

4. We define the "waiting time distribution", $P(t)$, as the probability for having observed no quantum jump up to time t and $w(t) = -dP/dt$. What is the physical interpretation of $w(t)$?

5. Compute $P(t)$ by dividing the time interval between 0 and t into $N \gg 1$ intervals of duration τ.

6. Show that $P(t)$ is then equal to the norm of the atomic state evolving from 0 to t under the influence of H_e. Compute $w(t)$. Comments?

We now consider the two-level system driven by a laser field, resonant on the $|g\rangle \to |e\rangle$ transition. The atom is initially in state $|g\rangle$. We assume that the mode of the field contains a very large coherent state, with $\overline{n}$ photons on the average.

7. Show qualitatively that for such large fields and short enough times and when we do not consider phenomena sensitive to the field phase, the coherent field can be approximated by an initial Fock state $|n\rangle$ (where n is $\overline{n}$ rounded to the closest integer).

8. Justify the form of the ion-field coupling Hamiltonian : $H_{if} = \hbar(\Omega_e/2)\big(|e, n-1\rangle\langle g, n| + \text{h.c.}\big)$ with self-explanatory notations. Show that spontaneous emission affects one of the states $\{|g, n\rangle, |e, n-1\rangle\}$ only. With which rate? To which state is the system driven in a spontaneous quantum jump?

9. Deduce that the evolution in the $\{|g, n\rangle, |e, n-1\rangle\}$ subspace under the action of spontaneous emission and of the laser drive is ruled, before the first quantum jump, by a simple non-hermitian Hamiltonian that will be written explicitly.

10. Writing the state as $c_e|e\rangle + c_g|g\rangle$ during this non-hermitian evolution, give an evolution equation for $c_e(t)$. Give the associated eigenfrequencies.

11. Solve the equation for a Rabi frequency Ω_e much larger than Γ_e. Justify, from the results of question 6, that $P(t) = |c_e|^2 + |c_g|^2$. Compute finally $P(t)$ and $w(t)$ and comment the physical signification of these results.

12. Examine also the case when $\Omega_e \ll \Gamma_e$. Show that $P(t)$ can then be approximated by a single exponential. Comment the physical signification of the associated time constant.

III. Three-level system

We now turn to the full three-level situation of the mercury ion. We assume here that the rates and Rabi frequencies obey the hierarchy $\Omega_e \gg \Gamma_e \gg \Gamma_f, \Omega_f$.

13. Show that the two laser beams (which both contain a large photon number and can be treated as Fock states for this discussion) couple $|g, n_e, n_f\rangle$ to two other atom-field states that will be given.

The energy of the state $|g, n_e, n_f\rangle$ is taken as the origin. We first treat the strong dressing of the $|g\rangle \to |e\rangle$ transition by the laser at exact resonance and leave level $|f\rangle$ apart for a while.

14. Show that the eigenstates of the atom-field hermitian Hamiltonian (dressed states) are $|+\rangle$ and $|-\rangle$, linear combinations of $|g, n_e, n_f\rangle$ and $|e, n_e - 1, n_f\rangle$. Give their explicit form and their energy ($|+\rangle$ will be chosen as the highest energy state).

15. Show that the dressed states are both prone to spontaneous emission with a rate $\Gamma_e/2$. Towards which states do they decay?

16. The laser at frequency ω driving the $|g\rangle \to |f\rangle$ transition is detuned by $\delta = \omega - \omega_f$ with respect to exact resonance. Show that, for an appropriate δ value, it induces a resonant coupling between $|+\rangle$ and $|f, n_e, n_f - 1\rangle$. Give a physical interpretation of this resonance condition. Show that the coupling of $|f, n_e, n_f - 1\rangle$ to state $|-\rangle$ is negligible in these conditions.

17. Give the associated atom-field coupling in the $\{|+\rangle, |f, n_e, n_f - 1\rangle\}$ basis. Show that the associated Rabi frequency is $\Omega_f/\sqrt{2}$. Write finally the total effective Hamiltonian including the non-hermitian part in the $\{|-\rangle, |+\rangle, |f, n_e, n_f - 1\rangle\}$ basis.

We compute the delay time distribution $P(t)$ between the emission of two successive photons by the atom. We take the emission of the first photon as the time origin. The atom is thus, immediately after this emission, in state $|g\rangle$. The result of the explicit resolution of the Schrödinger equation with this initial condition is $|\Psi(t)\rangle = c_+ |+\rangle + c_- |-\rangle + c_f |f, n_e, n_f - 1\rangle$, with

$$\begin{aligned} c_\pm(t) &= \pm\frac{1}{\sqrt{2}} e^{-\Gamma_e t/4} , \\ c_f(t) &= -i\frac{\Omega_f}{\Gamma_e}\left(e^{-\Gamma' t} - e^{-\Gamma_e t/4}\right) , \end{aligned}$$

where

$$\Gamma' = \frac{\Omega_f^2}{2\Gamma_e} + \frac{\Gamma_f}{2} .$$

18. Show that $P(t)$ can be written, at first significant order in the small quantities, as:

$$P(t) = (1-p)e^{-t/\tau_1} + pe^{-t/\tau_2} = P_{\rm short}(t) + P_{\rm long}(t) , \tag{1}$$

and give the expressions of τ_1, $\tau_2 \gg \tau_1$ and $p \ll 1$.

19. Give a physical interpretation of these terms, and in particular of the Ω_f^2/Γ_e contribution to Γ'. Give a numerical estimate of the relevant quantities for $\Omega_f = \sqrt{\Gamma_e \Gamma_f}$, a condition that we supposed fulfilled from now on.

20. We chose an arbitrary time T such that $\tau_1 \ll T \ll \tau_2$. We divide the intervals between two successive photon emissions in two sets. The "short" intervals correspond to a delay lower than T, the long ones to a delay larger than T. Compute the total probability for having a long time interval. Compute the average durations of long and short time intervals. Show that all these results are independent upon the choice of the delay threshold T.

21. Deduce from these results that there are two regimes in the emission, bright periods during which the fluorescence is intense and dark periods with no fluorescence.

22. Compute the average duration of a dark period, $\mathcal{T}_d$. Estimate with simple arguments the duration of a bright period $\mathcal{T}_b$. Compute the ratio $\mathcal{T}_d/\mathcal{T}_b$. Numerical values. Compare to the observations in Figure 1.

23. Guess what happens for other δ values. Draw qualitatively the behaviour of $\mathcal{T}_d/\mathcal{T}_b$ versus δ.

24. Show by qualitative arguments that the Optical Bloch Equations for this system admit a steady state. Where are the jumps? Explain the difference between the OBE predictions and the observed signals.

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Exam of December 18th, 2013

Jean-Michel Raimond, Julien Laurat, Hélène Perrin

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Preliminary

In 1999, Lene Hau and her group reported the observation of very slow light propagation in a cold sodium atoms medium (Nature, **397**, 594). By using Electromagnetically Induced Transparency, they were able to demonstrate the propagation of light pulses with velocities as low as 17 m/s, corresponding obviously to a very large index of refraction. Rather counterintuitively, this large refraction does not come at the expense of a large absorption.

This problem uses the formalism of the Maxwell-Bloch equations to describe this experiment. The first introductory part recalls standard results about absorption and dispersion in a two-level atom sample. The bulk of the problem then focuses on the experimental situation of a three-level Lambda structure addressed by two laser fields, a resonant intense pump field and a weak probe field, on which the slow propagation is observed.

I. Introduction

We consider the propagation of light in a sample of N two-level atoms per unit volume, with a ground state $|g\rangle$ (zero energy) and an excited state $|e\rangle$, with energy $\hbar\omega_e$. The $|g\rangle \rightarrow |e\rangle$ transition has a dipole matrix element d (considered, without loss of generality, to be real). The excited level $|e\rangle$ decays towards $|g\rangle$ with a spontaneous emission rate $\Gamma = \omega^3 d^2/3\pi\epsilon_0\hbar c^3$. The environment is at zero temperature. We will not consider here any other source of decoherence for the atomic system. The atomic state is described by the density matrix, which reads in the $\{|e\rangle, |g\rangle\}$ basis:

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}$$

It is driven by a classical probe laser field $E_s \exp(-i\omega t) + c.c.$ (Note that s here stands for 'sonde'). We will assume for the sake of simplicity that E_s is real and introduce the probe Rabi frequency $\Omega_s = dE_s/\hbar$. The probe detuning is $\delta = \omega_e - \omega$ (note that the sign convention for the detuning is not the same here – and in the lecture notes – as in the TD). The light propagates in the sample over a total length L. The light wavelength at resonance with the atomic transition is λ.

1. Recall, from the lecture notes, the optical Bloch equations for $\tilde{\rho}_{ee}$ and $\tilde{\rho}_{eg}$ in interaction representation with respect to the probe laser frequency.

2. We are interested in the weak probe regime, with an intensity far below the saturation. We thus set $\tilde{\rho} = \tilde{\rho}^{(0)} + \Omega_s\tilde{\rho}^{(1)}$ (note that, with this definition, $\tilde{\rho}^{(1)}$ and $\tilde{\rho}^{(0)}$ do not have the same dimension). The atoms are initially in state $|g\rangle$. Give $\tilde{\rho}^{(0)}$. Write the equation of evolution of $\tilde{\rho}_{eg}^{(1)}$ at the relevant order.

3. Give the steady state solution of this equation.

4. Give the density of polarization of the atomic medium, $\mathbf{P}$. Deduce that the electric susceptibility, $\chi = \chi' + i\chi''$, in the limit of a dilute atomic medium, is $\chi = \chi_0 2\Gamma/(2\delta - i\Gamma)$, where $\chi_0 = Nd^2/(\hbar\epsilon_0\Gamma)$. Give the expression of χ_0 as a function of N and λ. Give χ' and χ''. Under which condition is the dilute medium approximation valid?

5. Give in the same limit the index of refraction $n = n' + in''$. Plot qualitatively the variation of n' and n'' with δ.

6. Compute at resonance the characteristic length L_0 of the exponential decay of the light intensity. Give a simple expression of L_0^{-1} in terms of the resonant absorption cross section $\sigma = 3\lambda^2/2\pi$. Interpretation.

7. The optical density of the medium is defined as $OD = L/L_0$. What is its physical interpretation? Give an order of magnitude of L_0 and of the OD for Hau's experiment, based on the paper data: $L = 229$ μm; $\lambda = 589$ nm; $N = 3.10^{12}$ cm^{-3}; $\Gamma = 6.1\,10^7$ rad/s. Conclusions?

8. Give also an order of magnitude estimate of the maximum real part of the index of refraction. Are our hypotheses correct?

II. Three-level OBEs

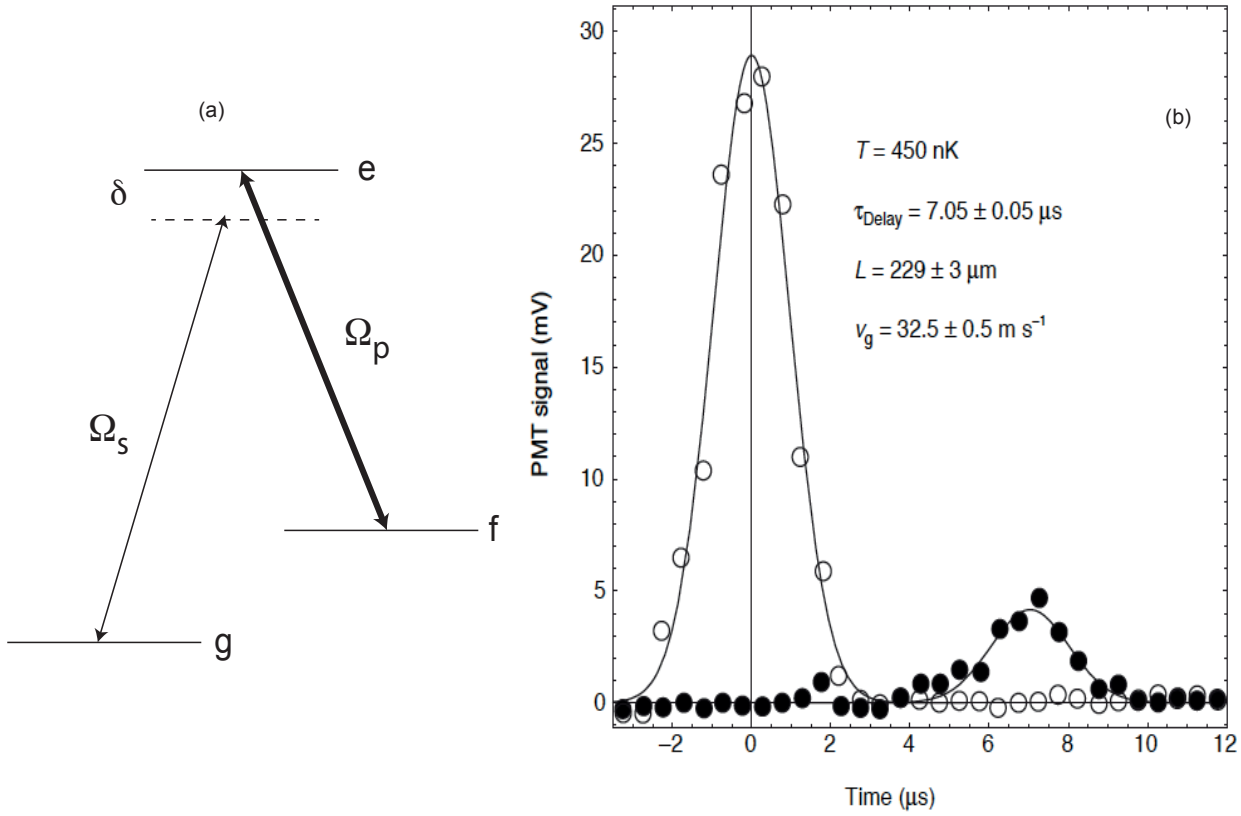

Figure 1: (a) Simplified level scheme. (b) experimental light propagation signal.

We now switch to the actual three-level configuration used in the paper (see figure 1). In addition to $|e\rangle$ and $|g\rangle$ (ground state with zero energy), we consider level $|f\rangle$, with energy $\hbar\omega_f$. In the experiment, $|g\rangle$ and $|f\rangle$ are two hyperfine levels of the sodium atom and can both be considered to have an infinite

lifetime. The $|f\rangle \to |e\rangle$ transition is driven by an 'intense' pump laser (frequency ω_p), with a Rabi frequency Ω_p (assumed to be real). The pump laser is assumed throughout this problem to be at exact resonance: $\omega_p = \omega_{ef} = \omega_e - \omega_f$. The $|g\rangle \to |e\rangle$ transition is driven by a weak 'probe' laser with the Rabi frequency Ω_s and the frequency $\omega_s = \omega_e - \delta$. The atom is initially in $|g\rangle$.

In the first part of this section, we consider the Hamiltonian equations of motion of the atomic density matrix. We then add the relaxation terms. Finally, we give a perturbative solution for the steady state. Along the calculation, we will focus only on the density matrix elements that are relevant for our discussion.

1. Show that, keeping only resonant and relevant terms, the full atomic Hamiltonian H can be put under the form:

$$H/\hbar = \omega_e |e\rangle\langle e| + \omega_f |f\rangle\langle f| - \frac{\Omega_s}{2}\left[e^{-i\omega_s t}|e\rangle\langle g| + \text{h.c.}\right] - \frac{\Omega_p}{2}\left[e^{-i\omega_p t}|e\rangle\langle f| + \text{h.c.}\right]$$

   Cast this in matrix form in the $\{|e\rangle, |f\rangle, |g\rangle\}$ basis. Note that we will not use interaction representation before question 6.

2. Deduce the equations for the Hamiltonian evolution of ρ_{eg} and ρ_{fg}. Is this set of equations closed?

3. We now proceed to include relaxation. What are a priori all the possible quantum jump operators compatible with our hypotheses?

4. In order to capture the essential physics without algebraic complications, we assume that $|e\rangle$ only decays towards $|g\rangle$ with the rate Γ. By a simple generalization of the two-level case, give the evolution equations for ρ_{eg} and ρ_{fg}.

5. As in the introductory part, we look for a perturbative solution of the Bloch equations for low probe powers, i.e. for a low value of Ω_s. We thus set: $\rho = \rho^{(0)} + \Omega_s \rho^{(1)}$ (same note as in question I2). Give $\rho^{(0)}$. Write the equations of motion of the relevant elements of $\rho^{(1)}$. Show that the resulting set of equations is closed

6. These equations contain an explicit time dependence. It can be removed by switching to an interaction representation. We thus set $\tilde{\rho}_{eg}^{(1)} = \rho_{eg}^{(1)} e^{i\omega_1 t}$ and $\tilde{\rho}_{fg}^{(1)} = \rho_{fg}^{(1)} e^{i\omega_2 t}$. Write the equations of motion of these new quantities. Which values of ω_1 and ω_2 make all coefficients in the evolution equations time-independent?

7. Show finally that the relevant equations of motion are the closed set:

$$\begin{aligned}
\frac{d\tilde{\rho}_{eg}^{(1)}}{dt} &= -(i\delta + \Gamma/2)\tilde{\rho}_{eg}^{(1)} + \frac{i}{2} + i\frac{\Omega_p}{2}\tilde{\rho}_{fg}^{(1)} \\
\frac{d\tilde{\rho}_{fg}^{(1)}}{dt} &= -i\delta\tilde{\rho}_{fg}^{(1)} + i\frac{\Omega_p}{2}\tilde{\rho}_{eg}^{(1)}
\end{aligned}$$

8. Give the stationary state solution of these two equations. Discussion.

III. Slow light propagation

We examine now the light propagation in the medium made up of the atoms described in the previous section.

1. By arguments similar to those of the introductory part, give the electric susceptibility χ of the medium for the probe field, as well as its real and imaginary parts. What happens for $\delta = 0$?

2. Give the inverse of the absorption length in these conditions, L_1^{-1}, as a function of L_0, Γ, δ, and Ω_p.

3. Recall, from the lecture notes, the aspect of the function $L_1^{-1}(\delta)$ in the case of a large pump intensity ($\Omega_p \gg \Gamma$). Give a physical interpretation.

4. Recall, from the lecture notes, the shape of $L_1^{-1}(\delta)$ in the other limit, $\Omega_p \ll \Gamma$. Is the name 'Electromagnetically Induced Transparency' reasonable for what happens?

5. Give a physical interpretation of the total absence of absorption at resonance.

6. Evaluate the full width at half maximum, Δ, of the transparency window in the $\Omega_p \ll \Gamma$ regime. We stay in this regime from now on. Discuss the $\Omega_p \to 0$ limit.

7. Give an approximate expression of L_1^{-1} as a function of δ near the center of the transparency window. We set rather arbitrarily as the transparence criterion that L_1^{-1} should be less than L^{-1}. Compute the allowed range of δ values as a function of Ω_p, Γ and OD, the optical density for the two-level situation (no pump field).

8. Give an approximate expression of the real part of the index of refraction n' near the centre of the transparency window.

9. In a medium with an index of refraction $n(\omega)$, the group velocity is $v_g = c/(n + \omega\, dn/d\omega)$. Compute the group velocity in the centre of the transmission window.

10. Show that the group velocity is close to the product of the two-level absorption length L_0 by the width Δ of the transparency window. Compare that with the speed of light and conclude.

11. Give an order of magnitude estimate of v_g in the conditions of Hau's experiment, with $\Omega_p = 3.6\,10^7$ rad/s. Compare this order of magnitude with the observation of figure 1. Are all our hypothesis fully satisfied in this regime?

12. Does the slow light phenomenon apply to arbitrary input wave packets? What are the conditions to observe it properly? What happens when they are not satisfied?

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Preliminary

This problem focuses on the resonant van der Waals interaction between two trapped atoms, in the context of Rydberg levels. Many groups are nowadays experimenting with Rydberg atoms for quantum simulation of many-body systems (see for instance Barredo et al. Phys. Rev. Lett., 2015, **114**, 113002). The basic principle of these experiments is to couple neighboring atoms in a lattice by the dipole-dipole interaction, which is particularly strong for these giant atomic orbitals. This interaction results in particular in an exchange process. When one atom is in the Rydberg level e and the other in the Rydberg level g (e and g are connected by a dipole transition), they can exchange their energy through the emission/absorption of a photon. This energy exchange, also called in other contexts Förster interaction, mimics a flip-flop interaction between two spin 1/2 systems. Complex evolution of coupled spin systems can thus be "quantum simulated" with chains of interacting Rydberg atoms.

This problem derives from a research project aiming at using trapped (circular) Rydberg atoms to implement such spin chains. It focuses on the complex interplay between the spin exchange interaction and the atomic motion in the trap. Of course the present problem will oversimplify some rather complex issues involved. Some results about a mechanical harmonic oscillator motion, as well as useful mathematical expressions and numerical values are given in the appendix section.

I. Introduction: van der Waals exchange interaction

We consider two two-level atoms separated by a distance L over the Ox axis. The atomic "ground" state is g and the excited state e, the transition frequency being $\omega_0/2\pi$. The two levels are linked by a dipole transition. Indices 1 and 2 will be used to distinguish the operators relevant to the two atoms. The atoms are represented as spin-1/2 systems, state e for atom i, $i \in \{1,2\}$, being identified with the $|+_i\rangle$ state along a fictitious OZ axis. The dipole matrix elements are $\mathbf{d}_1 = \mathbf{d}_2 = d\mathbf{u}_z$. where we assume, for the sake of simplicity, that the dipoles are aligned along the laboratory's Oz axis.

For numerical applications, we will $d = 1700\, ea_0$ (e: electron charge, a_0 Bohr radius), $\omega_0 = 2\pi \times 50$ GHz and $L = 10$ μm. The atoms have a mass $m = 1.4\,10^{-25}$ kg (rubidium atoms), which corresponds to $h/m = 4.6 \times 10^{-9}$ m^2·s^{-1} where h is the Planck constant. We use later $C_3 = d^2/(4\pi\varepsilon_0)$ with: $C_3/h = 2.8$ GHz·μm^3.

We study here the action of the dipole-dipole coupling between these two atoms. The interaction between two dipoles with this simple geometry is:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{d}_1 \cdot \mathbf{d}_2}{L^3}$$

1. Recall the expression of the quantum dipole operator of atom i as a function of d and of the Pauli matrices σ_j^i (where $j = X, Y, Z$) of the fictitious spin-1/2. We introduce the notation $W = d^2/(4\pi\varepsilon_0 L^3) = C_3/L^3$. Express V in terms of W and of the spin lowering and raising operators of both atoms $\sigma_\pm^{1,2}$.

2. We use the basis $\{|e,e\rangle, |e,g\rangle, |g,e\rangle, |g,g\rangle\}$ (atom 1 state is given first in the kets). Give the energies of these levels. Give the matrix of V in this basis in terms of W.

We observe that V has no contribution to first order to the energy of $|g,g\rangle$ or $|e,e\rangle$ and that V couples the degenerate states $|e,g\rangle$ and $|g,e\rangle$ to first order. As far as spin exchange is concerned, states $|g,g\rangle$ and $|e,e\rangle$ are mere spectator states and can be removed from the dynamics. From now on, we restrict the study to the subspace spanned by $|e,g\rangle$ and $|g,e\rangle$, whose energy is arbitrarily set to zero. We thus consider the two-atom system as another fictitious spin-1/2, with $|+\rangle = |e,g\rangle$ and $|-\rangle = |g,e\rangle$.

3. We put the Hamiltonian in the $\{|+\rangle, |-\rangle\}$ basis in the form $W\Sigma_X$. Explicit the matrix Σ_X. Give the eigenenergies $E_\pm = \pm\hbar\Omega_0/2$ and eigenvectors $|\Phi_\pm\rangle$ of V. Give the numerical value of $\Omega_0/2\pi$.

4. The atoms are initially prepared in $|e,g\rangle$. Give the explicit time evolution of the atomic state. Plot as a function of time the probability for finding atom 1 in e.

5. We consider the eigenenergies $E_\pm$ as potential energies for each eigenstate, function of the distance r between atoms, which may now differ from L. Deduce that the two atoms experience forces when they are in the $|\Phi_\pm\rangle$ eigenstates of V.

II. Exchange and motional coupling

We now treat the interplay between the exchange of energy between the atoms and their motion. We consider here only the one-dimensional motion along the axis Ox joining the two atoms. They are now trapped in two harmonic potentials centered at the points $x = 0$ and $x = L$, with a common oscillation frequency $\omega_x/2\pi = 2.5\,10^4$ Hz. For numerical applications, we take $L = 10$ μm. The positions of the atoms relative to the center of the respective traps will be noted x_1 and x_2, both in the classical and quantum contexts (these quantities are then operators). The spatial extension of the trap ground state is noted x_0 (see Appendix). The energy zero for the two oscillators is that of their ground state $|0\rangle$.

1. The force resulting from the dipole interaction between the atoms and their motion are treated classically. The atoms are in the $|\Phi_-\rangle$ eigenstate of the exchange hamiltonian V. Show by symmetry arguments that $x_1 = -x_2 = \Delta > 0$ at equilibrium.

2. Give the atomic distance at equilibrium as a function of L and Δ. Give the equilibrium position Δ by using the interatomic force at order zero in Δ/L. Numerical application: compute Δ, x_0/L and Δ/x_0 , where x_0 is the oscillators' ground state extension. Check that $\Delta \ll L$, justifying the zeroth order approximation. How are these results modified in the case of the $|\Phi_+\rangle$ eigenstate?

3. Give an expansion to first order in $x_{1,2}/L$ of the exchange potential. From now on, we will only use the expansion at this order. Show that the quantum exchange hamiltonian can be written as: $H_X = W\Sigma_X\,[1 + 3(x_1 - x_2)/L]$ ($W = C_3/L^3$).

4. Express H_X as a function of the annihilation and creation operators of the oscillators, $a_{1,2}$ and $a_{1,2}^\dagger$. Make use of $\eta = 3x_0/L$. What is the numerical value of η?

5. Show finally that the complete hamiltonian of the system is:

$$H = \hbar\omega_x \left(a_1^\dagger a_1 + a_2^\dagger a_2\right) + W\Sigma_X\,[1 - \eta\,(a_2 - a_1 + \text{h.c.})]\ .$$

6. The two atoms are assumed to be in the attractive state $|\Phi_-\rangle$. We approximate the ground state by a tensor product of $|\Phi_-\rangle$ and two coherent states with amplitudes $\alpha_{1,2}$. By analogy

with the classical case, justify qualitatively and briefly that the amplitudes are real and that $\alpha_2 = -\alpha_1$. Give the average value of H in this state as a function of α_1 and find the minimum of this quantity (variational method to find the ground state). Compare the results with the classical computations of the introductory part. Same questions when the atoms are in $|\Phi_+\rangle$.

III. The thick of it: complete quantum dynamics

We now tackle the full dynamical problem and give a complete analytical solution.

1. In order to solve the Schrödinger equation, we use a time-independent unitary transform U, setting $|\Psi(t)\rangle = U\left|\widetilde{\Psi}(t)\right\rangle$. Remind the Schrödinger equation for $\left|\widetilde{\Psi}(t)\right\rangle$ and the corresponding hamiltonian $\tilde{H}$ in terms of U and H.

2. Justify briefly, based on the previous results, why we choose now $U = D_1(\beta_1\Sigma_X)D_2(\beta_2\Sigma_X)$ where we define $D_i(A) = \exp(Aa_i^\dagger - A^\dagger a_i)$, reducing to the normal displacement operator when A is a complex number.

3. Using the Baker-Hausdorff lemma, compute $D_i(-\beta_i\Sigma_X)a_iD_i(\beta_i\Sigma_X)$. Deduce without further calculation the value of $D_i(-\beta_i\Sigma_X)a_i^\dagger D_i(\beta_i\Sigma_X)$ and the expressions of $U^\dagger a_i U$ and $U^\dagger a_i^\dagger U$.

4. Give explicitly the new hamiltonian $\tilde{H}$.

5. Show that, for a proper choice of the displacement amplitudes $\beta_1 = -\beta_2 = \beta$ (β is real), $\tilde{H}$ reduces, within a change of the energy origin, into the hamiltonian of two atoms undergoing an exchange together with two uncoupled independent harmonic oscillators. Give the eigenstates of $\tilde{H}$. Compare β to the previous results.

6. The initial state is $|\Psi(0)\rangle = |+,0,0\rangle$ (the two atoms are in e and g and the oscillators are at their uncoupled equilibrium position). Show that the corresponding initial transformed state is $\left|\widetilde{\Psi}(0)\right\rangle = (1/\sqrt{2})\left(|\Phi_+\rangle\,|-\beta\rangle\,|\beta\rangle + |\Phi_-\rangle\,|\beta\rangle\,|-\beta\rangle\right)$.

7. Use the previous results to give the explicit time dependence of $|\Psi(t)\rangle$. For the sake of simplicity, let $\gamma = \beta(1 - \exp(-i\omega_x t))$.

8. Discuss the time evolution of $|\Psi(t)\rangle$ and the entanglement of the atomic state with the motion.

9. By tracing the global density matrix $|\Psi(t)\rangle\langle\Psi(t)|$ over the motional states (useful formula in appendix), compute as a function of time the probability $P(t)$ for finding the atomic state in its initial state $|e,g\rangle$. Show that, when $\beta \gg 1$, the oscillations predicted in the first part are washed out, but periodically revive. Give the amplitude C of the oscillations as a function of time. Plot qualitatively $P(t)$ with the typical numerical parameters of the model.

10. Define a quality factor Q for the exchange oscillation as the number of observable oscillations. Taking simple criteria to properly define the relevant quantities, write the value of Q in the limit of a large β amplitude as a function of η.

11. How does Q depend upon the experimental parameters? How can we make it the largest?

12. Facultative question. Discuss, at least qualitatively, the case of three atoms in a row. Do the simple methods used here apply?

IV. Appendix: mechanical quantum oscillator

We recall here some key results on the quantum treatment of a mechanical oscillator with a resonant frequency ω_x. The ground state extension is $x_0 = \sqrt{\hbar/(2m\omega_x)}$, the position operator is defined as $x = x_0(a+a^\dagger)$. In the frame of this problem, the hamiltonian is $H = \hbar\omega_x\, a^\dagger a$. All the other conclusions of field quantization applied to one mode apply directly.

The Baker Hausdorff lemma writes:

$$e^A a e^{-A} = a + [A, a] + \frac{1}{2!}[A, [A, a]] + \dots \ .$$

For two coherent states with amplitudes α and β, $\mathrm{Tr}(|\alpha\rangle\langle\beta|) = \langle\beta\,|\alpha\rangle$.