Cosmological Stochastic Gravitational-Wave Backgrounds

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Stochastic gravitational-wave background (SGWB) is an incoherent superposition of many uncorrelated gravitational-wave (GW) sources. It could be cosmological: for example, vacuum fluctuations from the early universe could produce a stochastic background. It could also be astrophysical: for example, adding contributions from all binary black hole coalescences in the universe would produce a stochastic background. While this background is expected to be permanent (i.e. not transient), it is not expected to have a predictable waveform. However, different models predict different power spectra of the stochastic background, and possible different distributions across the sky and different polarizations.

We describe the SGWB in terms of the normalized energy density of GWs:

$$\Omega_{GW} \quad = \quad \frac{1}{\rho_{c,0}} \frac{d\rho_{GW}}{d \ln f} \tag{1}$$

where we think of $d\rho_{GW}$ as the energy density in the frequency band between f and $f + df$, and $\rho_{c,0}$ is the present-day critical energy density needed to close the universe:

$$\rho_{c,0} \quad = \quad \frac{3H_0^2 c^2}{8\pi G}. \tag{2}$$

Here, H_0 is the present-day value of the Hubble parameter (capturing the rate of expansion of the universe), c is the speed of light and G is Newton's constant. Note that this definition is a similar (but not identical) to the definition to the normalized energy densities typically used in cosmology.

Since this definition involves GW frequencies, it is helpful to convert our metric perturbation into the frequency domain. Specifically, we write the following decomposition:

$$h_{ab}(t,\vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int d\hat{\Omega} h_A(f,\hat{\Omega}) e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} e^{A}_{ab}(\hat{\Omega}) \tag{3}$$

This is effectively a Fourier transform in both time and space, coupled with the decomposition into the plus and cross polarizations. The unit vector $\hat{\Omega}$ is a direction on the 2-D sphere (sky), described by two angles (θ,ϕ), from which the GW is arriving. We can therefore write the wavevector as $\vec{k} = 2\pi f\hat{\Omega}/c$. The amplitudes (at a given frequency, from a given direction in the sky) obey $h_A(f,\hat{\Omega}) = h^*_A(-f,\hat{\Omega})$, which is a consequence of the fact that h_{ab} is real. The polarization tensors have to be defined relative to the wave propagation direction, which is $\hat{\Omega}$. Specifically, we can define them as follows (see Allen-Romano paper):

$$e^{+}_{ab} = \hat{m}_a\hat{m}_b - \hat{n}_a\hat{n}_b \tag{4}$$

$$e^{\times}_{ab} = \hat{m}_a\hat{n}_b - \hat{n}_a\hat{m}_b \tag{5}$$

$$\hat{\Omega} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{6}$$

$$\hat{m} = \sin\phi\hat{x} - \cos\phi\hat{y} \tag{7}$$

$$\hat{n} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \tag{8}$$

At this point, we have to make some assumptions about the SGWB. It is common to assume that different polarizations are uncorrelated, and that different frequencies are uncorrelated - while this is certainly not required, most if not all SGWB models obey these assumptions. It is also typical to assume that the SGWB is isotropic so that different sky directions are not correlated - there certainly are cases where this assumption breaks down (for example, galactic SGWB will not be isotropic). We will proceed with these assumptions and write

$$\langle h^*_A(f,\hat{\Omega}) h_{A'}(f',\hat{\Omega}')\rangle = \delta_{AA'}\delta(f-f')\delta^2(\hat{\Omega},\hat{\Omega}')H(f) \tag{9}$$

$$\delta(\hat{\Omega},\hat{\Omega}') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta') \tag{10}$$

where $H(f)$ can be thought of as the GW strain power spectrum in frequency. We will further assume that the SGWB is gaussian, so that $\langle h_A(f,\hat{\Omega})\rangle = 0$.

We can then proceed with the calculation of the SGWB spectrum:

$$
\begin{aligned}
\Omega_{GW} &= \frac{1}{\rho_{c,0}} \frac{d\rho_{GW}}{d\ln f} && (11)\\
&= \frac{f}{\rho_{c,0}} \frac{c^4}{32\pi G} \frac{d}{df} \langle \dot{h}_{ab} \dot{h}^{ab} \rangle && (12)\\
&= \frac{2c^4 f}{32\pi G \rho_{c,0}} \sum_{AA'} \int df' d\hat{\Omega} d\hat{\Omega}' \langle h^*_A(f, \hat{\Omega}) h_{A'}(f', \hat{\Omega}') \rangle \\
&\quad (-2\pi i f) e^{-2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} 2\pi i f' e^{2\pi i f'(t-\hat{\Omega}'\cdot\vec{x}/c)} e^A_{ab}(\hat{\Omega}) e^{ab}_{A'}(\hat{\Omega}') && (13)\\
&= \frac{c^4 f \pi}{4G\rho_{c,0}} \sum_{AA'} \int df' d\hat{\Omega} d\hat{\Omega}' \delta_{AA'} \delta(f-f') \delta^2(\hat{\Omega}, \hat{\Omega}') H(f) \\
&\quad f f' e^{-2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} e^{2\pi i f'(t-\hat{\Omega}'\cdot\vec{x}/c)} e^A_{ab}(\hat{\Omega}) e^{ab}_{A'}(\hat{\Omega}') && (14)\\
&= \frac{c^4 f \pi}{4G\rho_{c,0}} \sum_{A} \int d\hat{\Omega} H(f) \\
&\quad f^2 e^{-2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} e^{2\pi i f(t-\hat{\Omega}\cdot\vec{x}/c)} e^A_{ab}(\hat{\Omega}) e^{ab}_{A}(\hat{\Omega}) && (15)\\
&= \frac{4c^4 f 4\pi^2}{4G\rho_{c,0}} f^2 H(f) && (16)\\
&= \frac{4\pi^2 c^4}{G} \frac{8\pi G}{3H_0^2 c^2} f^3 H(f) && (17)\\
&= \frac{32\pi^3 c^2}{3H_0^2} f^3 H(f) && (18)\\
& && (19)
\end{aligned}
$$

In the third step, the factor of 2 in the numerator comes from the fact that the integral over f (which is annulled by the derivative wrt f) has the range between $-\infty$ and $+\infty$, while in line 3 we only consider positive frequencies (so $\int_{-\infty}^{+\infty} = 2\int_0^\infty$). So the SGWB energy density is proportional to the strain power spectrum, but note the f^3 factor which effectively weighs different frequency bins of the power spectrum. Because of this f^3 factor, detectors operating at lower frequencies (with the same strain sensitivity) would have better sensitivity to Ω_{GW}.

As noted above, there are multiple potential sources of the SGWB, and we will now consider some of the most interesting ones. We start with the inflationary model. As we discussed in the cosmology part of the class, inflation gives a rapid expansion of the universe, so that quantum fluctuations become expanded to macroscopic scales. These quantum fluctuations are thought to give rise to the CMB anisotropy, and eventually to lead to the formation of structure we observe today. However, these quantum fluctuations could also exist today as gravitational waves.

To see what happens to quantum modes during expansion, let us consider the example of a quantum harmonic oscillator, that is a pendulum of length l and angular frequency $\omega = \sqrt{g/l}$. Quantum mechanically, this system is described by eigenstates involving the Hermite polynomials:

$$\psi_n(x) = \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}} e^{-\alpha^2 x^2/2} H_n(\alpha x) \tag{20}$$

$$\alpha = \sqrt{\frac{m\omega}{\hbar}} \tag{21}$$

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right) \tag{22}$$

Consider the vacuum state $n = 0$ for some initial length of the pendulum l_i:

$$\psi_0(x) = \sqrt{\frac{\alpha_i}{\sqrt{\pi}}} e^{-\alpha_i^2 x^2/2} \tag{23}$$

$$\alpha_i = \sqrt{\frac{m\omega_i}{\hbar}} \tag{24}$$

$$\omega_i = \sqrt{\frac{g}{l_i}} \tag{25}$$

$$E_0 = \frac{\hbar\omega}{2} \tag{26}$$

Now let us lower the pendulum to some final length l_f corresponding to the classical frequency of $\omega_f = \sqrt{g/l_f}$. The new quantum state of the pendulum will depend on how quickly we lower the pendulum. If we lower it slowly, so that $\Delta t >> \omega_f^{-1}$, then the quantum state will continuously morph from the original ground state into the ground state corresponding to the new length. This would be an adiabatic process, and the number of quanta (in this case zero) would be conserved. However, if we lower the pendulum quickly, $\Delta t << \omega_i^{-1}$, then the quantum state will not have time to change. As a result, the quantum state of the pendulum would stay the same, but it would no longer be the ground state. Like any state, this state can be expanded in terms of the new eigenstates of the system:

$$\psi_n^f(x) = \sqrt{\frac{\alpha_f}{2^n n! \sqrt{\pi}}} e^{-\alpha_f^2 x^2/2} H_n(\alpha_f x) \tag{27}$$

$$\alpha_f = \sqrt{\frac{m\omega_f}{\hbar}} \tag{28}$$

$$\psi_0^i(x) = \sum_{n=0}^{\infty} c_n \psi_n^f(x) \tag{29}$$

$$c_n = \int_{-\infty}^{\infty} \psi_0^{i\dagger}(x) \psi_n^f(x) dx \tag{30}$$

The coefficients in the last line can be computed analytically, but we won't bother (see C&A, page 182). The energy level of this state after the pendulum is lowered is given by

$$\langle E \rangle = \int_{-\infty}^{\infty} \psi_0^{i\dagger}(x) \hat{H} \psi_0^{i\dagger}(x) dx = \sum_{n=0}^{\infty} c_n^2 E_n^f \tag{31}$$

where $\hat{H}$ is the Hamiltonian of the system. The last sum can be done analytically giving

$$\langle E \rangle = \hbar\omega_f \left[\frac{(\omega_i - \omega_f)^2}{4\omega_i \omega_f} + \frac{1}{2} \right] \tag{32}$$

In other words, the fast lowering of the pendulum (expansion) resulted in production of

$$N = \frac{(\omega_i - \omega_f)^2}{4\omega_i \omega_f} \rightarrow_{l \to \infty} \frac{\omega_i}{4\omega_f} \tag{33}$$

quanta of energy! In quantum field theory, a field acts as an infinite collection of quantum oscillators. As the space expands, the field expands with it, which has an equivalent effect to lengthening a pendulum. In the case of a gravitational quantum field, this would result in the production of gravitons, whose wavelength would expand with the inflating universe to macroscopic scales (turning them into gravitational waves). Hence, we expect the inflationary process to result in a stochastic background of gravitational waves—stochastic because the original gravitons were generated as (we expect) Gaussian fluctuations in the vacuum.

To see this, we follow the Maggiore review paper (see pg 57 and later). We assume $\hbar = c = 1$ unless noted otherwise. Specifically, we start with the FRW metric, but we rewrite it in terms of the conformal time $d\eta = dt/a(t)$:

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2) \tag{34}$$

In the presence of GWs, we modify the spatial part of this metric:

$$g_{\mu\nu} = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}) \tag{35}$$

where $\eta_{\mu\nu} = [-1, 1, 1, 1]$ is the flat metric and in the TT gauge the perturbation can be expanded as:

$$h_{ab}^{TT} = \sqrt{8\pi G} \sum_A \sum_{\vec{k}} \phi_{\vec{k}}^A(\eta) e^{i\vec{k}\cdot\vec{x}} e_{ab}^A(\hat{\Omega}) \tag{36}$$

As usual, x represents the comoving coordinates, so the physical coordinates are $x_{phys} = ax$. Similarly, the wavevectors represent the comoving momentum, so the physical momentum would be given by $\vec{k}_{phys} = \vec{k}/a$. Note that we have also separated the time and spatial dependence of the perturbation metric. This is because the time-dependence is no longer a simple sinusoidal (typical for GW), i.e. it has to include the expansion in time as well.

With the metric defined, we go back to the Einstein's equation, which we try to linearize. Assuming that the spacetime is described by some background metric $g^{(0),\mu\nu}$, the Einstein equation can be reduced to (see C&A):

$$0 = g^{(0),\mu\nu} \nabla_\mu^{(0)} \nabla_\nu^{(0)} \bar{h}_{ab} \tag{37}$$

where the index (0) denotes the unperturbed metric, ∇ denotes covariant derivatives, and we only keep terms up to the order $O(R^{-1})$, where R is the curvature of the background (FRW) metric. We can therefore write (in TT gauge):

$$g^{(0),\mu\nu} \nabla_\mu^{(0)} \nabla_\nu^{(0)} h_{ab} = g^{(0),\mu\nu} \nabla_\mu^{(0)} \left(\frac{\partial h_{ab}}{\partial x^\nu} - \Gamma_{\nu a}^{(0)\gamma} h_{\gamma b} - \Gamma_{\nu b}^{(0)\gamma} h_{a\gamma} \right) \tag{38}$$

$$\Gamma_{\alpha\beta}^{(0)\gamma} = \frac{1}{2} g^{(0)\gamma\delta} \left(\frac{\partial g_{\beta\delta}^{(0)}}{\partial x^\alpha} + \frac{\partial g_{\delta\alpha}^{(0)}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}^{(0)}}{\partial x^\delta} \right) \tag{39}$$

Note that the background metric is only dependent on time, so the only terms in the connection that survive will be the time derivatives, which obey (we will denote derivatives relative to η with primes): $g'_{\mu\nu} = \frac{2a'}{a} g_{\mu\nu}$. Hence these terms will look like $2a'h_{ab}/a$. Applying the second covariant derivative to the first term will give:

$$\frac{\partial^2 h_{ab}}{\partial x_\mu \partial x^\mu} = \sum_{A,\vec{k}} \left(\phi''_{\vec{k}} + k^2 \phi_{\vec{k}}\right) e^{i\vec{k}\cdot\vec{x}} e^A_{ab} \tag{40}$$

Applying it to the second term yields terms where the ordinary (time) derivative is applied to $2a'h_{ab}/a$, which gives $2a'h'_{ab}/a \sim 2a'\phi'/a$ (other terms involving a'' and a'^2 cancel). Doing the calculation carefully gives

$$\phi''_{\vec{k}} + 2\frac{a'}{a}\phi'_{\vec{k}} + k^2 \phi_{\vec{k}} = 0 \tag{41}$$

[*** We include a more detailed derivation of this equation due to Andrew Matas: We start with the FRW metric, but we rewrite it in terms of the conformal time $d\eta = dt/a(t)$:

$$ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2) \tag{42}$$

In the presence of GWs, we modify the spatial part of this metric

$$ds^2 = a^2(\eta)(-d\eta^2 + (\delta_{ij} + h_{ij})dx^i dx^j) \tag{43}$$

where h_{ij} is spatially traceless and transverse

$$g^{(0),ij} h_{ij} = \nabla^{(0),i} h_{ij} = 0 \tag{44}$$

where i, j refer to spatial indices only, for example $i = \{1, 2, 3\}$. Note that we have not written the most general metric perturbation. In general, on FRW we can decompose the metric perturbation into pieces that have special transformation properties under spatial rotations (scalars, vectors, and tensors). We have included only tensor perturbations, which correspond to gravitational waves. The terms we are missing are the scalar and vector perturbations. Their dynamics decouple from the tensor modes to the order we are working, meaning that it is consistent for us to treat the tensor modes separately A complete discussion of all the different kinds of perturbations can be found in section 2 of the review by Mukhanov et al, http://luth2.obspm.fr/IHP06/lectures/silk-uzan/IHP_bib/bmf.pdf.

We can plug this perturbation into Einstein's equations. We find after a straightforward but tedious calculation

$$
\begin{aligned}
\delta G_0^0 &= 0 \\
\delta G_i^0 &= 0 \\
\delta G_j^i &\propto \frac{\partial^2 h_j^i}{\partial \eta^2} + 2\frac{a'}{a} h_j'^i - \delta_{kl} \frac{\partial^2 h_j^i}{\partial x_k \partial x_l}
\end{aligned} \tag{45}
$$

where $G_\nu^\mu = R_\nu^\mu - \frac{1}{2} R \delta_\nu^\mu$. The symbol δ is an instruction to perturb and work to linear order in h_j^i.

We can gain some insight into where these equations come from without going through the full derivation.

- The first equation $G_0^0 = 0$ by symmetry. One way to see this is that the tensor perturbation h_j^i has two free indices, but the equation has none. The only way h_j^i can appear is either contracted with itself, like h_i^i, or contracted with background covariant derivatives, like $\nabla^{(0),i} \nabla_j^{(0)} h_i^j$. But both of these vanish using the conditions above. The second equation $G_0^i = 0$ for similar reasons. The only way h_j^i can appear is as $\nabla_i^{(0)} h_j^i$, but this is zero.

- To understand the last equation, first note that every term has two derivatives. The first term has two time derivatives acting on h_j^i, the second term has one derivative acting on the background metric (a') and one derivative acting on h_j^i, and the last term has two (spatial) derivatives acting on h_j^i. Since the background metric only depends on time, any derivative acting on the background metric must be a time derivative, such as $g'_{\mu\nu} = \frac{2a'}{a} g_{\mu\nu}$. We can understand the terms with two derivatives on h_j^i by looking more closely at the Ricci tensor. Driven by the intuition that the free indices should sit on h_j^i, the non-vanishing terms in Einstein's equations are

$$
\begin{aligned}
\delta G_j^i &= \delta R_j^i = g^{(0),i\rho} \delta R_{\rho j} \\
&= g^{(0),i\rho} \frac{\partial \delta \Gamma_{\rho j}^\mu}{\partial x^\mu} + g^{(0),i\rho} \Gamma_{\mu\lambda}^{(0),\mu} \delta \Gamma_{\rho j}^\lambda \\
&\quad - g^{(0),i\rho} \Gamma_{j\mu}^{(0),\lambda} \delta \Gamma_{\rho\lambda}^\mu - g^{(0),i\rho} \Gamma_{\rho\mu}^{(0),\lambda} \delta \Gamma_{j\lambda}^{(0),\mu} + \delta g^{i\rho} R_{\rho j}^{(0)} \\
&= g^{(0),i\rho} \nabla_\mu^{(0)} \delta \Gamma_{\rho j}^\mu + \delta g^{i\rho} R_{\rho j}^{(0)}
\end{aligned} \tag{46}
$$

Going from the first line to the second line is not immediately obvious, but it does follow by writing out the definition of δR_{ij} and demanding that all free indices sit on h^i_j. Going from the second line to the last line follows from noticing that the background connections $\Gamma^{(0)\lambda}_{\mu\nu}$ repackage into a background covariant derivative.

Let's expand the first term on the second line of the previous equation to build our intuition:

$$\begin{aligned}
g^{(0),i\rho}\frac{\partial\delta\Gamma^{\mu}_{\rho j}}{\partial x^{\mu}} &= \frac{1}{a^2}\eta^{i\rho}\frac{\partial}{\partial x^{\mu}}\left(-\frac{1}{2}g^{(0),\mu\nu}\frac{\partial}{\partial x^{\nu}}(a^2 h_{\rho j})\right)\\
&= -\frac{1}{2a^2}\left(\eta^{\mu\nu}\frac{\partial h^i_j}{\partial x^{\mu}\partial x^{\nu}} - 2\frac{a'}{a}h'^i_j + 2\left(\frac{a'^2}{a^2}-\frac{a''}{a}\right)h^i_j\right)
\end{aligned}$$

We see the second derivatives on h^i_j, as well as derivatives on the background metric (which of course are time derivatives).

Being careful with the other terms, and adding everything together leads to Equation 45. Somewhat miraculously, all the terms $\sim a''$, $(a')^2$ cancel.

Given the equations for the metric perturbation, we would like to understand the gravitational waves. We then introduce the mode expansion

$$h_{ij} = \sqrt{8\pi G}\sum_A\sum_{\vec{k}}\phi^A_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}}e^A_{ij}(\hat{\Omega}) \tag{47}$$

Using the mode expansion, we can write the first term as

$$\frac{\partial^2 h_{ij}}{\partial x_{\mu}\partial x^{\mu}} = \sum_{A,\vec{k}}\left(\phi''_{\vec{k}} + k^2\phi_{\vec{k}}\right)e^{i\vec{k}\cdot\vec{x}}e^A_{ij} \tag{48}$$

Using this, Einstein's equation $\delta G^i_j = 0$ becomes

$$\phi''_{\vec{k}} + 2\frac{a'}{a}\phi'_{\vec{k}} + k^2\phi_{\vec{k}} = 0 \tag{49}$$

***]

It is also convenient to define a new variable and rewrite the last equation in terms of it:

$$\psi_{\vec{k}} = a\phi_{\vec{k}} \tag{50}$$

$$\psi''_{\vec{k}} + \left(k^2 - \frac{a''}{a}\right)\psi_{\vec{k}} = 0 \tag{51}$$

Now, we suppose there are two regimes, I and II, of cosmological evolution, with a transition between them on some time scale ΔT. For example, these two regimes could be inflation and radiation domination, or radiation domination and matter domination. Suppose the solutions to Equation 51 are $f_{\vec{k}}(\eta)$ and $F_{\vec{k}}(\eta)$ in the phases I and II respectively. We can then decompose the metric as:

$$\begin{aligned} h_{ab}^{TT} &= \sqrt{8\pi G}\sum_A \int \frac{d^3k}{(2\pi)^3\sqrt{2k}}\frac{1}{a(\eta)} \\ &\quad \left[a_A(\vec{k})f_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + a_A^\dagger(\vec{k})f_{\vec{k}}^*(\eta)e^{-i\vec{k}\cdot\vec{x}}\right] e_{ab}^A(\hat{\Omega}) \end{aligned} \quad (52)$$

where $a_A(\vec{k}), a_A^\dagger(\vec{k})$ are the creation and annihilation operators for the mode $\vec{k}$ in phase I:

$$a_+(\vec{k})|0\rangle_I = a_\times(\vec{k})|0\rangle_I = 0 \quad (53)$$

Similarly, in the phase II we have the decomposition:

$$\begin{aligned} h_{ab}^{TT} &= \sqrt{8\pi G}\sum_A \int \frac{d^3k}{(2\pi)^3\sqrt{2k}}\frac{1}{a(\eta)} \\ &\quad \left[A_A(\vec{k})F_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + A_A^\dagger(\vec{k})F_{\vec{k}}^*(\eta)e^{-i\vec{k}\cdot\vec{x}}\right] e_{ab}^A(\hat{\Omega}) \end{aligned} \quad (54)$$

$$A_+(\vec{k})|0\rangle_{II} = A_\times(\vec{k})|0\rangle_{II} = 0 \quad (55)$$

Since both f, f^* and F, F^* are complete sets of functions (basis sets), we can express one in terms of the other. This relation is known as the *Bogoliubov tranformation*:

$$F_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} = \sum_{\vec{k}'}\left(\alpha_{\vec{k}\vec{k}'}f_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}} + \beta_{\vec{k}\vec{k}'}f_{\vec{k}'}^*e^{-i\vec{k}'\cdot\vec{x}}\right) \quad (56)$$

Inserting this into Equation 54 gives

$$
\begin{aligned}
h_{ab}^{TT} &= \sqrt{8\pi G}\sum_{A\vec{k}'}\int \frac{d^3k}{(2\pi)^3\sqrt{2k}}\frac{e_{ab}^A(\hat{\Omega})}{a(\eta)} \\
&\left(A_A(\vec{k})\left(\alpha_{\vec{k}\vec{k}'}f_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}} + \beta_{\vec{k}\vec{k}'}f_{\vec{k}'}^{*}e^{-i\vec{k}'\cdot\vec{x}}\right) + A_A^{\dagger}(\vec{k})\left(\alpha_{\vec{k}\vec{k}'}^{*}f_{\vec{k}'}^{*}e^{-i\vec{k}'\cdot\vec{x}} + \beta_{\vec{k}\vec{k}'}^{*}f_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}}\right)\right) \\
&= \sqrt{8\pi G}\sum_{A\vec{k}'}\int \frac{d^3k}{(2\pi)^3\sqrt{2k}}\frac{e_{ab}^A(\hat{\Omega})}{a(\eta)} \\
&\left((A_A(\vec{k})\alpha_{\vec{k}\vec{k}'} + A_A^{\dagger}(\vec{k})\beta_{\vec{k}\vec{k}'}^{*})f_{\vec{k}'}e^{i\vec{k}'\cdot\vec{x}} + (A_A^{\dagger}(\vec{k})\alpha_{\vec{k}\vec{k}'}^{*} + A_A(\vec{k})\beta_{\vec{k}\vec{k}'})f_{\vec{k}'}^{*}e^{-i\vec{k}'\cdot\vec{x}}\right) \\
&= \sqrt{8\pi G}\sum_{A\vec{k}'}\int \frac{d^3k}{(2\pi)^3\sqrt{2k}}\frac{e_{ab}^A(\hat{\Omega})}{a(\eta)} \\
&\left((A_A(\vec{k}')\alpha_{\vec{k}'\vec{k}} + A_A^{\dagger}(\vec{k}')\beta_{\vec{k}'\vec{k}}^{*})f_{\vec{k}}e^{i\vec{k}\cdot\vec{x}} + (A_A^{\dagger}(\vec{k}')\alpha_{\vec{k}'\vec{k}}^{*} + A_A(\vec{k}')\beta_{\vec{k}'\vec{k}})f_{\vec{k}}^{*}e^{-i\vec{k}\cdot\vec{x}}\right)
\end{aligned} \tag{57}
$$

where in the last line we swapped (renamed) $k \leftrightarrow k'$ and we swapped the summation and integration operations. Comparing with Eq. 52 we see

$$a_A(\vec{k}) = \sum_{\vec{k}'}(A_A(\vec{k}')\alpha_{\vec{k}'\vec{k}} + A_A^{\dagger}(\vec{k}')\beta_{\vec{k}'\vec{k}}^{*}) \tag{58}$$

and similarly

$$A_A(\vec{k}) = \sum_{\vec{k}'}(a_A(\vec{k}')\alpha_{\vec{k}\vec{k}'}^{*} - a_A^{\dagger}(\vec{k}')\beta_{\vec{k}\vec{k}'}^{*}) \tag{59}$$

If the cosmological background metric is only time dependent but isotropic and homogeneous, then the gravitational field cannot change the momentum of particles, just energy. In this case, we can write

$$\alpha_{\vec{k}\vec{k}'} = \alpha_f\delta_{\vec{k}\vec{k}'} \tag{60}$$

$$\beta_{\vec{k}\vec{k}'} = \beta_f\delta_{\vec{k}\vec{k}'} \tag{61}$$

$$A_A(\vec{k}) = \sum_{\vec{k}'}(a_A(\vec{k}')\alpha_f^{*}\delta_{\vec{k}\vec{k}'} - a_A^{\dagger}(\vec{k}')\beta_f^{*}\delta_{\vec{k}\vec{k}'}) \tag{62}$$

$$A_f = a_f\alpha_f^{*} - a_f^{\dagger}\beta_f^{*} \tag{63}$$

where in the last line we suppressed the polarization index A and labeled the raising and lowering operators with the frequency rather than wavevector. Bogoliubov coefficients satisfy additional relations—the one that is relevant for us is

$$\sum_{\vec{k}} \left(\alpha_{\vec{k}_1 \vec{k}} \alpha^*_{\vec{k}_2 \vec{k}} - \beta_{\vec{k}_1 \vec{k}} \beta^*_{\vec{k}_2 \vec{k}} \right) = \delta_{\vec{k}_1 \vec{k}_2} \tag{64}$$

$$\sum_{\vec{k}} \left(\alpha_f^2 \delta_{\vec{k}_1 \vec{k}} \delta_{\vec{k}_2 \vec{k}} - \beta_f^2 \delta_{\vec{k}_1 \vec{k}} \delta_{\vec{k}_2 \vec{k}} \right) = \delta_{\vec{k}_1 \vec{k}_2} \tag{65}$$

$$\alpha_f^2 - \beta_f^2 = 1 \tag{66}$$

In the second line we assumed the background metric is isotropic and homogeneous, and in the last line we assumed $\vec{k}_1 = \vec{k}_2$. Now, suppose we have some state with occupation numbers $\{n_f\}$ in phase I, relative to the number operators $a_f^\dagger a_f$. Then,

$$a_f^\dagger a_f |\{n_f\}\rangle = n_f |\{n_f\}\rangle \tag{67}$$

Suppose the universe expands rapidly (inflation), and consider the modes for which the transition is too fast for the state to adjust: $2\pi f \Delta T << 1$. Since the state does not change, we can now evaluate the occupation numbers in phase II, using the number operator $A_f^\dagger A_f$. So,

$$\begin{aligned}
N_f &= \langle\{n_f\}| A_f^\dagger A_f |\{n_f\}\rangle \\
&= \langle\{n_f\}| (a_f^\dagger \alpha_f - a_f \beta_f)(a_f \alpha_f^* - a_f^\dagger \beta_f^*) |\{n_f\}\rangle \\
&= \langle\{n_f\}| \alpha_f^2 a_f^\dagger a_f + \beta_f^2 a_f a_f^\dagger - \alpha_f^* \beta_f a_f a_f - \alpha_f \beta_f^* a_f^\dagger a_f^\dagger |\{n_f\}\rangle \\
&= \langle\{n_f\}| \alpha_f^2 a_f^\dagger a_f + \beta_f^2 a_f a_f^\dagger - \alpha_f^* \beta_f a_f a_f - \alpha_f \beta_f^* a_f^\dagger a_f^\dagger |\{n_f\}\rangle \\
&= \alpha_f^2 n_f + \beta_f^2 (n_f + 1) \\
&= (\beta_f^2 + 1) n_f + \beta_f^2 (n_f + 1) \\
N_f &= n_f + 2\beta_f^2 \left(n_f + \frac{1}{2} \right)
\end{aligned} \tag{68}$$

where in the third line the last two terms give zero expectation values since the operators change the occupancy levels in the ket vector, which is then orthogonal to the bra vector. Also, we have used the standard relations for number operators:

$$a_f^\dagger a_f |\{n_f\}\rangle = n_f |\{n_f\}\rangle \tag{69}$$

$$a_f a_f^\dagger |\{n_f\}\rangle = (n_f + 1) |\{n_f\}\rangle \tag{70}$$

What we observe is that the occupancy level has been boosted by a factor of $1 + 2\beta_f^2$. Furthermore, even the ground state corresponding to $n_f = 0$ is amplified by $2\beta_f^2$. In other words, the transition from one phase to the other increases the vacuum fluctuation modes, at frequencies for which $2\pi f \Delta T << 1$. If we set the transition time to be set by the Hubble parameter at the time, $\Delta T \sim H^{-1}$, then $2\pi/\lambda << H$ or equivalently $\lambda/(2\pi) >> H^{-1}$. In other words, if the wavelength of the mode is larger than the horizon size at the time, it will get amplified; otherwise there is no amplification for sub-horizon size modes.

Note that inflation usually erases the information from the previous era. This is, however, not true with the vacuum fluctuations: the occupancy level in phase II, N_f, is very sensitive to the occupancy level in phase I, n_f.

We next compute the Bogoliubov coefficient for the specific example of the transition from inflation to radiation dominated phase. During inflation, the scale factor exponentially increases with time, $a(t) = Ce^{Ht}$, with H constant during inflation, so in conformal time we have:

$$d\eta = \frac{dt}{a} = \frac{1}{C} e^{-Ht} dt \tag{71}$$

$$\eta = -\frac{1}{CH} e^{-Ht} = -\frac{1}{Ha} \tag{72}$$

$$a(\eta) = -\frac{1}{H\eta} \tag{73}$$

This lasts until the radiation domination starts at some time t_1 (or η_1). During RD ($\eta > \eta_1$) we know $a = Ct^{1/2}$, so we have

$$d\eta = \frac{dt}{a} = \frac{dt}{Ct^{1/2}} \tag{74}$$

$$\eta - \eta_1 = \frac{2}{C} t^{1/2}|_{t_1}^{t} = \frac{2}{C^2} Ct^{1/2}|_{t_1}^{t} = \frac{2a(t)}{C^2} - \frac{2a(t_1)}{C^2} = \frac{2a(t)}{C^2} + \frac{2}{H\eta_1 C^2} \tag{75}$$

$$a(\eta) = \left(\eta - \eta_1 - \frac{2}{H\eta_1 C^2}\right) \frac{C^2}{2} \tag{76}$$

$$a'(\eta) = \frac{C^2}{2} = \frac{1}{H\eta^2}|_{\eta_1} = \frac{1}{H\eta_1^2} \tag{77}$$

$$a(\eta) = \left(\eta - \eta_1 - \frac{H\eta_1^2}{H\eta_1}\right) \frac{1}{H\eta_1^2} = \frac{1}{H\eta_1^2} (\eta - 2\eta_1) \tag{78}$$

We have matched a and a' at the boundary $\eta = \eta_1$, which fixed the coefficients. With these forms for $a(\eta)$, we can solve the differential equation in

Eq. 51 in the two phases. The solutions are:

$$\psi^I_{\vec{k}}(\eta) = \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} \tag{79}$$

$$\psi^{II}_{\vec{k}}(\eta) = \alpha_k e^{-ik\eta} + \beta_k e^{ik\eta} \tag{80}$$

Matching the solution at the boundary, $\psi_I(\eta_1) = \psi_{II}(\eta_1)$, $\psi'_I(\eta_1) = \psi'_{II}(\eta_1)$ gives

$$\alpha_k = 1 - \frac{i}{k\eta_1} - \frac{1}{2k^2\eta_1^2} \tag{81}$$

$$\beta_k = \frac{1}{2k^2\eta_1^2} \tag{82}$$

We immediately see that if we are starting with a vacuum state, $n_k = 0$, then after the transition into the second (radiation dominated) phase the number of quanta will be (see Eq. 68)

$$N_k = \beta_k^2 = \frac{1}{4k^4\eta_1^4} \tag{83}$$

We now try to relate this to the quantities we measure today, such as the frequency f of the mode and the physical momentum (wavenumber) $k_{phys} = k/a$. We can write (using 0 to denote present day values)

$$k|\eta_1| = k_{phys,0}a_0|\eta_1| = 2\pi f a_0|\eta_1| = \frac{2\pi f a_0}{Ha(t_1)} \tag{84}$$

$$= \frac{2\pi f}{H}\left(\frac{t_0}{t_{eq}}\right)^{2/3}\left(\frac{t_{eq}}{t_1}\right)^{1/2} \tag{85}$$

where we have used the phase I (inflation) relation $a(t_1) = a(\eta_1) = -1/(H\eta_1)$, where H is the Hubble parameter at the time of inflation. In the last line, we used the known evolution of the scaling factor $a(t)$ during the matter dominated phase ($\sim t^{2/3}$ up to the radiation-matter equality) and radiation dominated phase ($\sim t$, after radiation-matter equality). Setting $z_{eq} \approx (t_0/t_{eq})^{2/3}$ we can define

$$f_1 = \frac{H}{2\pi z_{eq}}\left(\frac{t_1}{t_{eq}}\right)^{1/2} \tag{86}$$

$$k|\eta_1| = \frac{f}{f_1} \tag{87}$$

$$N_f = \frac{1}{4k^4\eta_1^4} = \frac{f_1^4}{4f^4} \tag{88}$$

Since in radiation domination $H(t) = 1/(2t)$, at the transition time t_1 between inflation and RD we will have $t_1 = 1/(2H)$, where H is the Hubble parameter value during inflation. Inserting numerical values and choosing $H = 10^{-4}M_{planck}$ gives:

$$f_1 \approx 10^9 \left(\frac{H}{10^{-4}M_{planck}}\right)^{1/2} \text{Hz} \tag{89}$$

Next, let's compute the energy density given this number of modes. To do this, we use the usual statistical argument: if we have $n(\vec{x},\vec{k})$ gravitons per cell of phase space (which in our case $= N_f$, since in the isotropic and homogeneous universe only the magnitude $|\vec{k}| = k = 2\pi f$ matters), then the energy density will be given by (recall there are two polarizations, and recall $E = \omega\hbar = 2\pi f$)

$$\rho_{GW} = 2\int \frac{d^3k}{(2\pi)^3} N_f 2\pi f = 4\pi \int d^3 f N_f f \tag{90}$$

$$= 16\pi^2 \int f^3 N_f df = 16\pi^2 \int f^4 N_f d(\ln f) \tag{91}$$

$$\Omega_{GW}(f) = \frac{1}{\rho_{c,0}}\frac{d\rho_{GW}}{d\ln f} = \frac{16\pi^2 f^4 N_f}{\rho_{c,0}} \tag{92}$$

$$\Omega_{GW}(f) = \frac{4\pi^2 f_1^4}{\rho_{c,0}} = \frac{4\pi^2 \hbar f_1^4}{\rho_{c,0}c^3} \tag{93}$$

$$\sim 10^{-13}\left(\frac{H}{10^{-4}M_{planck}}\right)^2 \tag{94}$$

So we see that the SGWB energy spectrum is flat, i.e. frequency independent and also rather small in amplitude. This is the contribution of the inflation-RD transition, and it applies to all frequencies such that $2\pi f << 1/\Delta T = H$ at the time of inflation-RD transition. To compute what this means in terms of present day frequencies, recall the entropy argument:

$$g(T_*)a^3(t_*)T_*^3 = g(T_0)a_0^3T_0^3 \tag{95}$$

$$f_0 = f_*\frac{a(t_*)}{a_0} = f_*\left(\frac{g(T_0)}{g(T_*)}\right)^{1/3}\frac{T_0}{T_*} \approx 10^{-13} f_* \frac{1\ \text{GeV}}{T_*} \tag{96}$$

$$f_0 < \frac{10^{-13}H}{2\pi}\frac{1\ \text{GeV}}{T_*} \sim 10^{-14}10^{-4}\frac{M_{planck}c^2}{\hbar}\frac{1\ \text{GeV}}{T_*} \tag{97}$$

$$f_0 < 10^9\ \text{Hz} \tag{98}$$

where we have assumed that the degrees-of-freedom parameter is $g(T_0) \approx 3$ today and $g(T_*) \approx 100$ at the time of inflation (when all particles were relativistic), that $T_0 = 2.725$ K today and that $T_* = 10^{-4} M_{planck}$ at the time of the transition. So this SGWB spectrum is flat up to 10^9 Hz at the level of $\sim 10^{-13}$.

We know, of course, that there will be another transition later in the universe, namely the RD-MD equality. This transition will result in another amplification of modes with frequencies $f < f_{eq} \sim 10^{-16}$ Hz. We can solve for the Bogoliubov coefficients associated with this transition similarly to what we have done before, to get the spectrum:

$$\Omega_{GW}(f) \approx 10^{-13} \left(\frac{f_{eq}}{f}\right)^2 \left(\frac{H}{10^{-4} M_{planck}}\right)^2 \tag{99}$$

This range of the spectrum is also cut off at lower frequencies: the mode has to be inside the horizon today, which implies $f > H_0 \sim 3 \times 10^{-18}$ Hz. We also note that in many inflationary models the inflation is driven by a scalar field rolling down a very flat (but still sloped) potential. In these models, the Hubble parameter is not constant which results in a slight (model-dependent) tilt in the spectrum: $\Omega_{GW} \sim f^{n_T}$.

We note that there are several other cosmological models of SGWB (see GW slides):

- The inflationary mechanism discussed above is based on the amplification of vacuum fluctuation. However, we currently know very little about inflation, and it is possible that there are other mechanisms for production of GWs. For example, at the end of inflation the energy from the inflaton must be transferred to the standard model particles, and potentially gravitons. In some (preheating) models this can happen via a resonance, resulting in an efficient production of GWs, and causing a peak in the SGWB spectrum (above the flat spectrum discussed above) typically at high frequencies. Other models include back-reaction: as the inflaton produces other (eg gauge) particles, these particles can produce the inflaton field and extend inflation, resulting also in an increase in the SGWB at high frequencies. An example of this is the axion-inflaton model, studied by Peloso and colleagues. Depending on the model details, some of these models may be within reach of current and future terrestrial GW detectors.

- There are also more exotic models of the early universe. For example, the pre-big-bang models assume that the universe started off large, contracted and went through something like a singularity (big bang), after which the standard cosmology follows. While the physics in such models is speculative, one can go through the same steps of amplifying the vacuum modes as the universe transitions between various phases to compute the SGWB spectrum. One can then use SGWB measurements to constrain these models (rule out parts of the parameter space).

- It is also possible that there were additional phases in the universe, before radiation and after inflation. If they were characterized by stiff equation of state ($w > 1/3$), then they could also result in amplification of the SGWB, at relatively high frequencies. Buonanno and Boyle have studied this problem in a general setting, and have shown that GW detectors have the unique ability to probe these early phases in the evolution of the universe.

- As we discussed before, the universe could have gone through phase transitions as it cooled (and as various symmetries broke). The transitions would nucleate at various points in the universe, and bubbles of new phase would start to grow around the nucleating points. As the bubbles collide they would break spherical symmetries and cause GWs, or they could perturb the matter inside the bubbles which in turn could generate GWs. The SGWB spectrum from this mechanism would be peaked around a characteristic frequency, typically determined by the duration of the phase transition (which in turn is related to the energy scale (i.e. temperature) at which the transition happens). For the electroweak phase transition (when the EW symmetry was broken), the energy scale is about 100 GeV - 1 TeV, resulting in the characteristic frequency today of about 1 mHz, which is in the frequency band of eLISA. If other symmetries are broken at higher energies, they would correspond to higher frequencies and potentially be in the LIGO band. In either case, the amplitude of the SGWB in these models is typically low, and likely out of reach of either LIGO or eLISA.

- We mentioned cosmic strings in the past as 1D topological defects that may have formed during the phase transitions in the early universe. They are a generic prediction of quantum field theories describing the early universe, and they are also predicted by string theory (fundamental strings inflated to cosmological scales). Kinks and cusps on cosmic string loops would create bursts of GWs, and if we sum up all such bursts we obtain a stochastic GW background. This summation has to assume some model for cosmic string distribution in the universe, and there are numerical models that try to simulate evolution of such networks. The calculation has been done, and depending on the model large fractions of the parameter space have already been ruled out by different experiments (LIGO, indirect bounds due to BBN and CMB, pulsar timing observations).