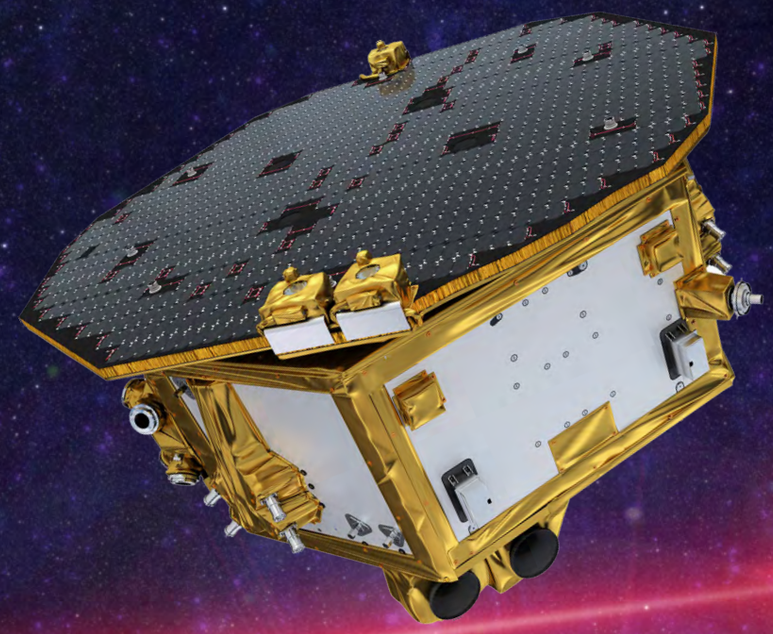
Albert Einstein Institute
Max Planck Institute for Gravitational Physics and
Leibniz Universität Hannover

Space based Gravitational Wave Observations

LISA and its pathfinder



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Institut für Gravitationsphysik
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Hannover

Les Houches
July 2018

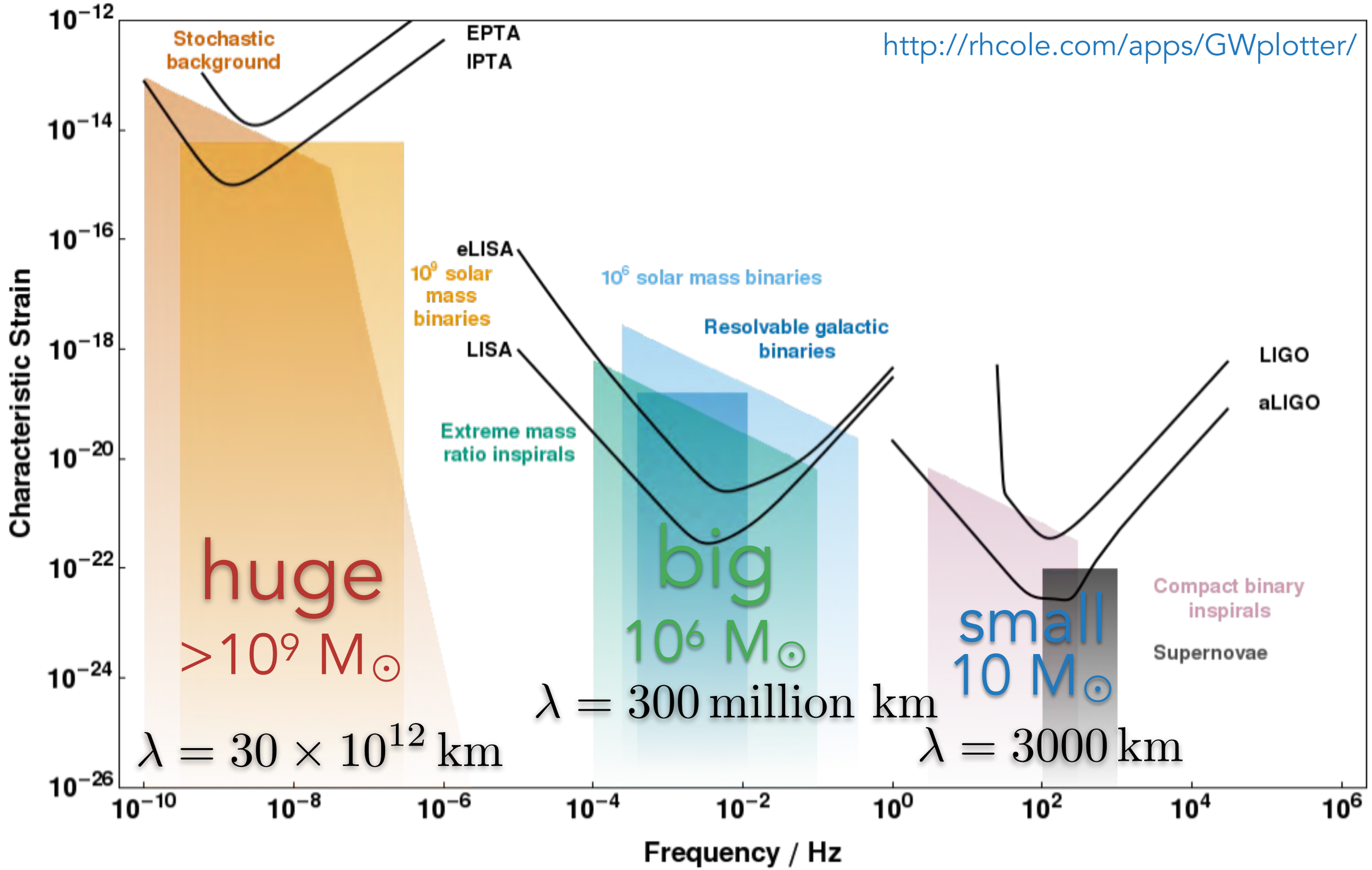
Overview of lecture(s)

- Why low(er) frequency?
- Low frequency, compare to ground based
- Measurement principle for LISA
- Discussion of noise sources
- TDI: the basics
- Building the LISA noise curve and science performance
- LISA Pathfinder: what did we test, and why?

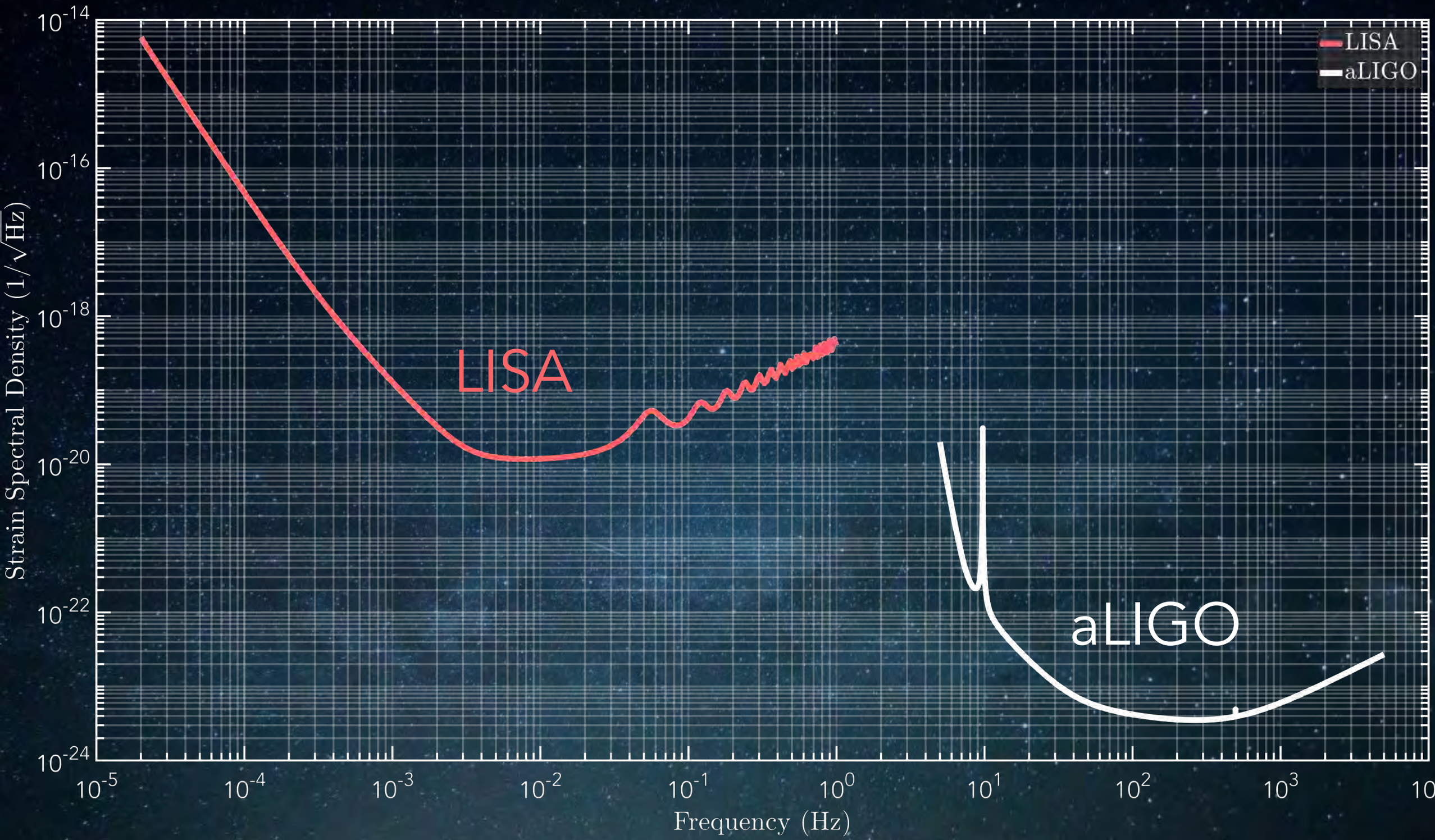
Martin Hewitson: Who he?

- Undergrad and PhD in Glasgow
- At AEI in Hannover since 2004
- Worked on:
  - GEO600 calibration and commissioning
  - LISA Pathfinder Data Analysis and operations
  - LISA mission formulation and instrumentation

GW Spectrum


http://rhcole.com/apps/GWplotter/
Characteristic Strain
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
10^{-24}
10^{-26}
Stochastic background
EPTA
IPTA
eLISA
10^9 solar mass binaries
10^6 solar mass binaries
Resolvable galactic binaries
LISA
Extreme mass ratio inspirals
LIGO
aLIGO
Compact binary inspirals
Supernovae
huge
$>10^9\ M_\odot$
$\lambda = 30 \times 10^{12}$ km
big
$10^6\ M_\odot$
$\lambda = 300$ million km
small
$10\ M_\odot$
$\lambda = 3000$ km
10^{-10} 10^{-8} 10^{-6} 10^{-4} 10^{-2} 10^{0} 10^{2} 10^{4} 10^{6}
Frequency / Hz

Compare space and ground strain



Challenges of observing at low frequency

- What limits ground based detectors at low frequency?
- Isolation:
  - suspensions on ground, free-fall in space
- What limits at low frequencies?
  - on ground: external displacements, external forces
  - in space: small forces on test masses (which are typically smaller, less massive)
- Interferometry challenges
  - temperature, thermo-mechanical stability, dynamic range
  - readout noise

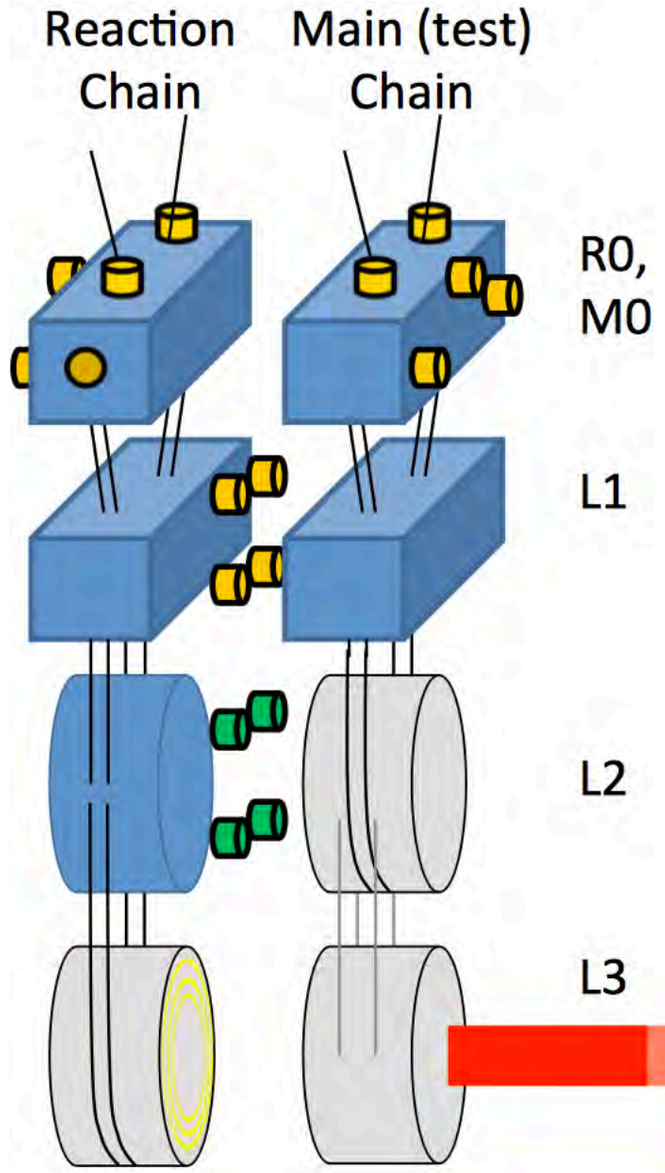
Remember the goal

- want to measure the relative proper motion of **free-falling** test masses

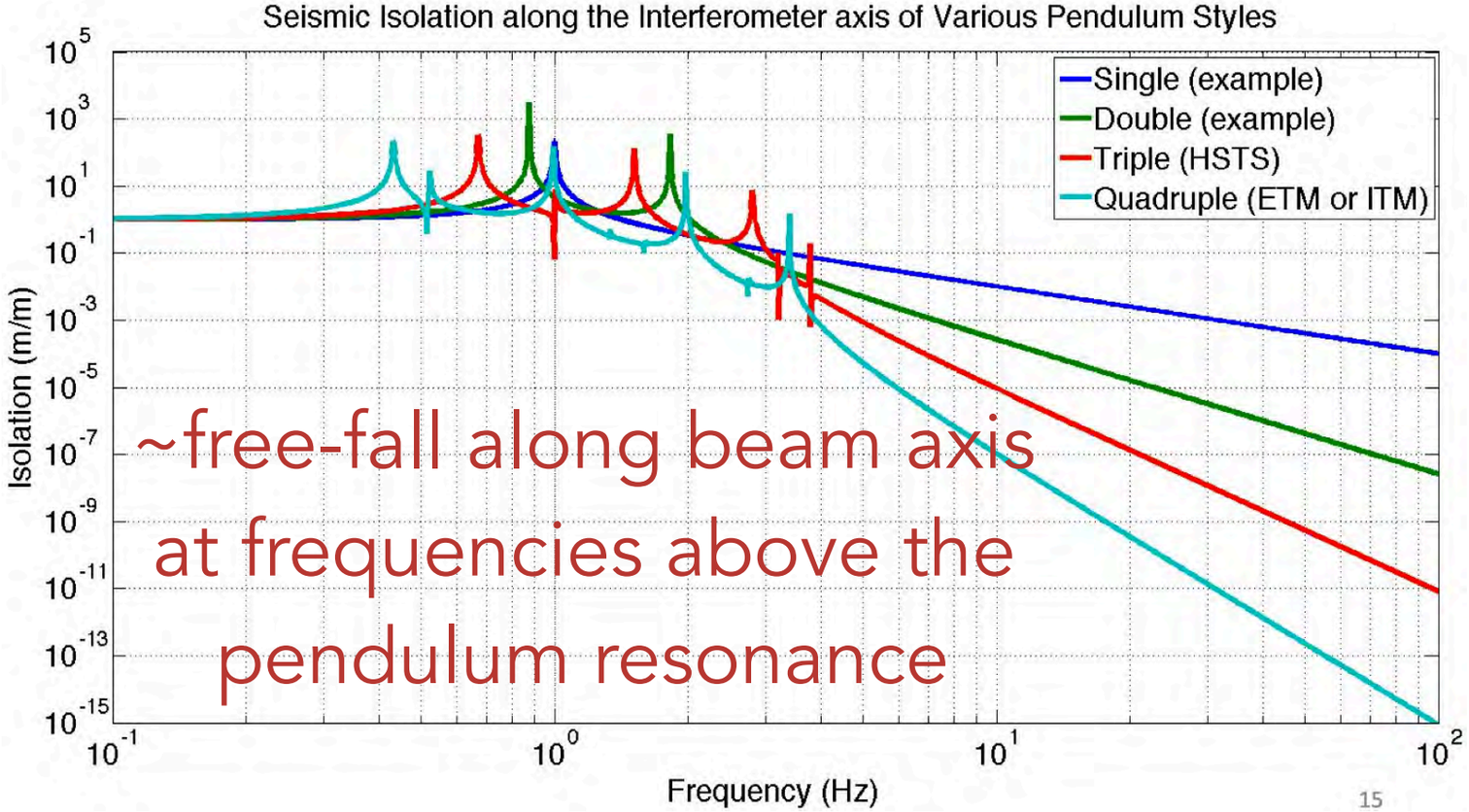
- infer from that the effect of gravitational waves

Isolation on ground

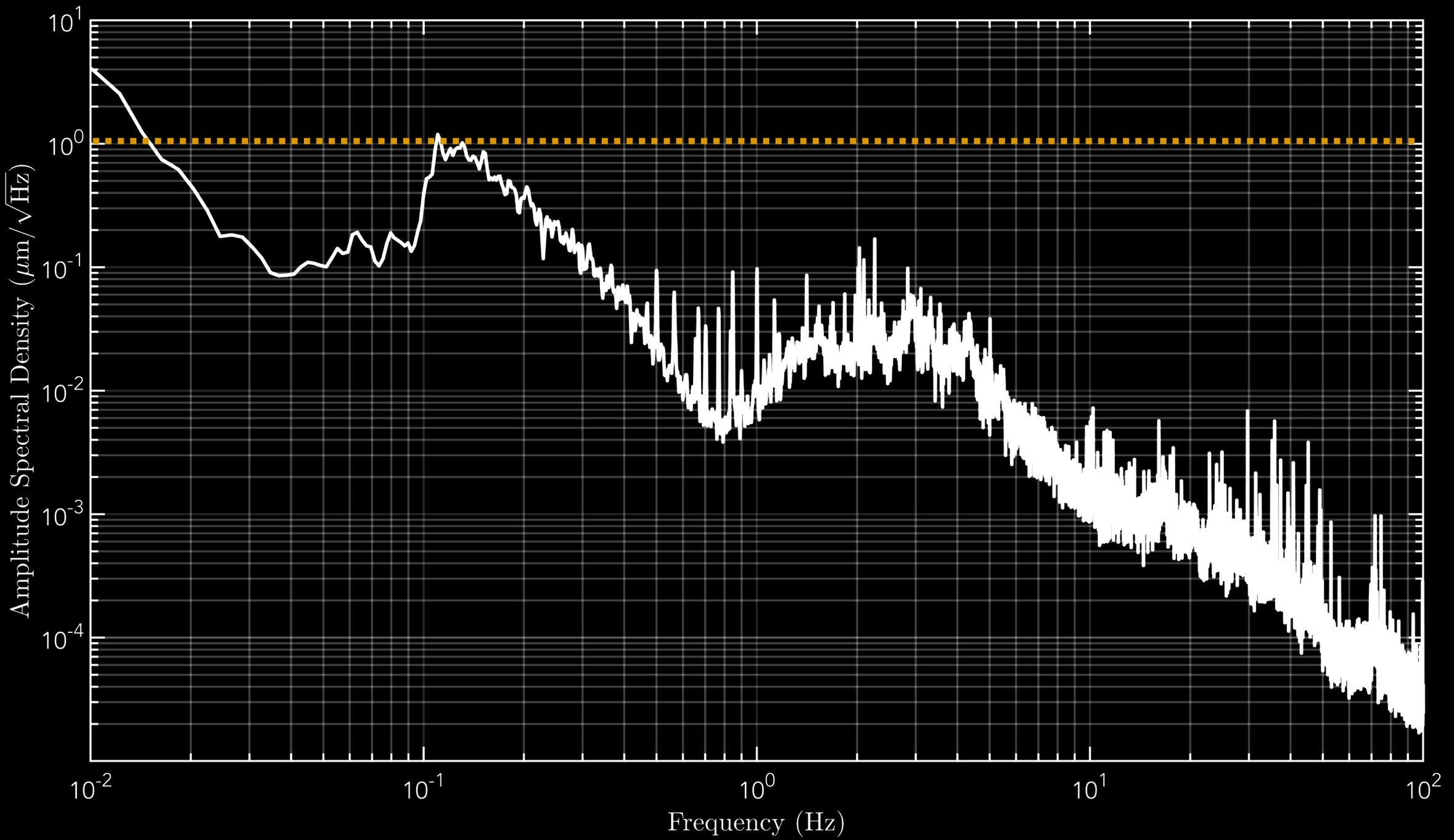
- Complex pendulum style isolation for test masses
- Get $1/f^2$ isolation from each stage



Brett Shapiro, G1400964-v3

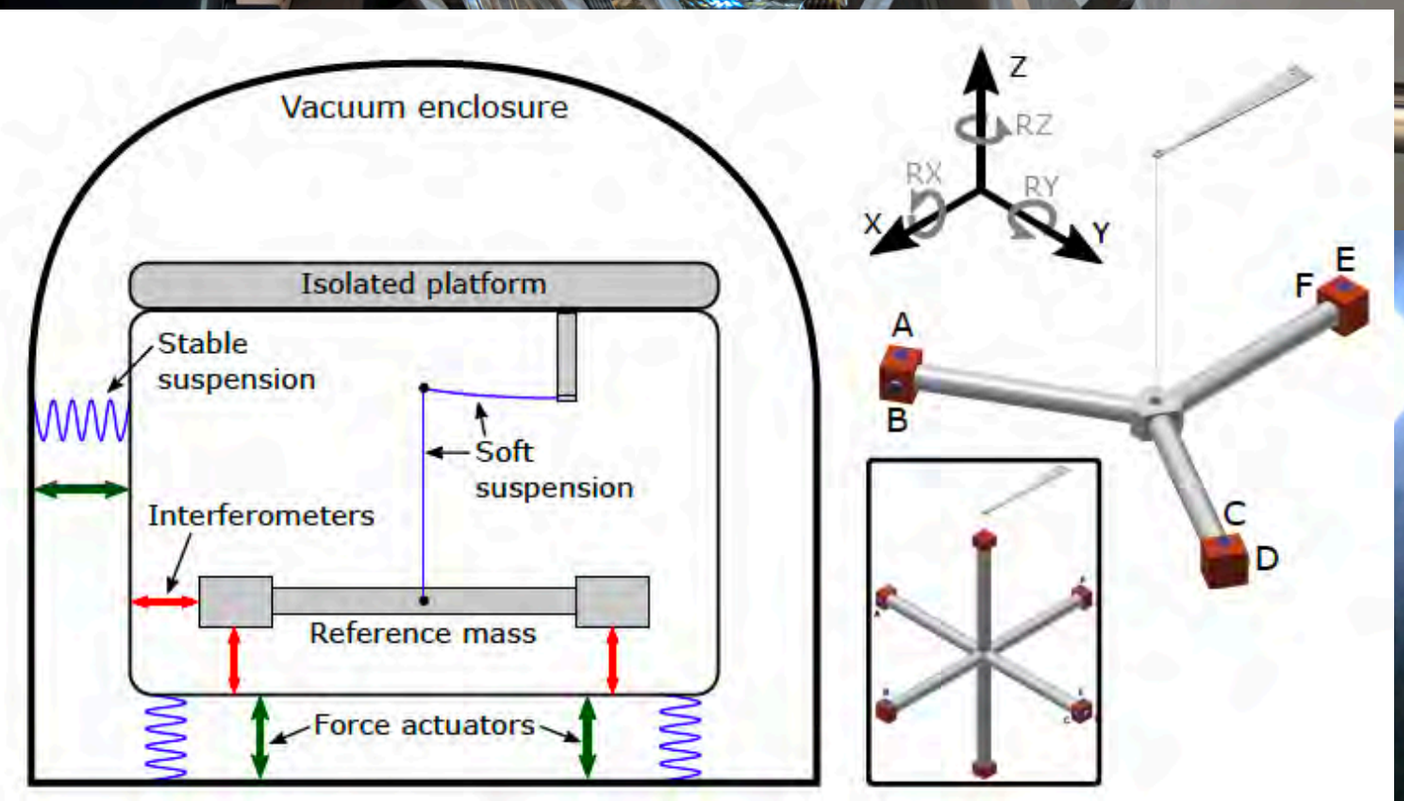


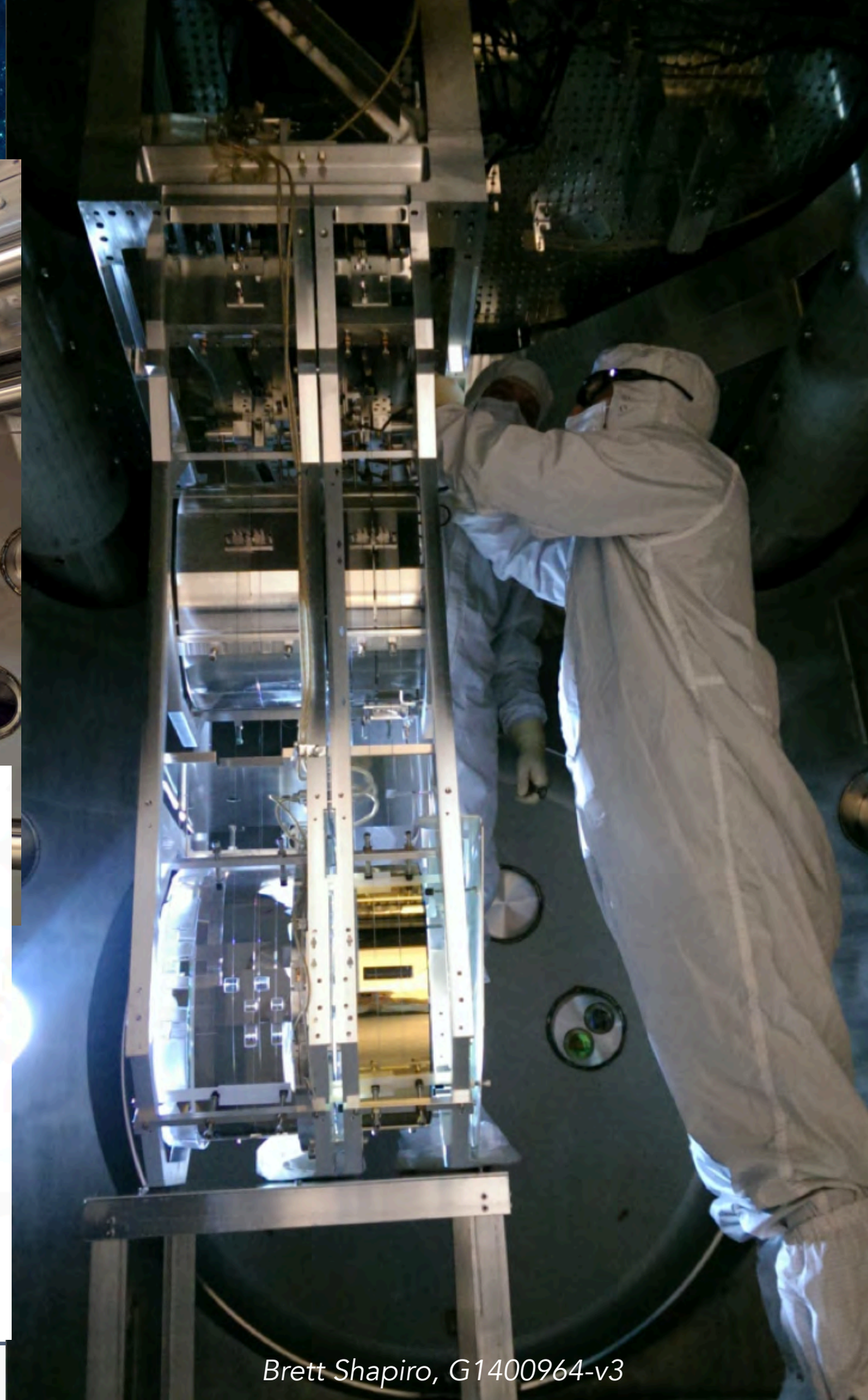
Typical ground motion



LTPDA 3.0.14.dev (R2017b), 2018-04-19 08:25:32.181 UTC, ltpda: 55ef5a9, evaluateCode

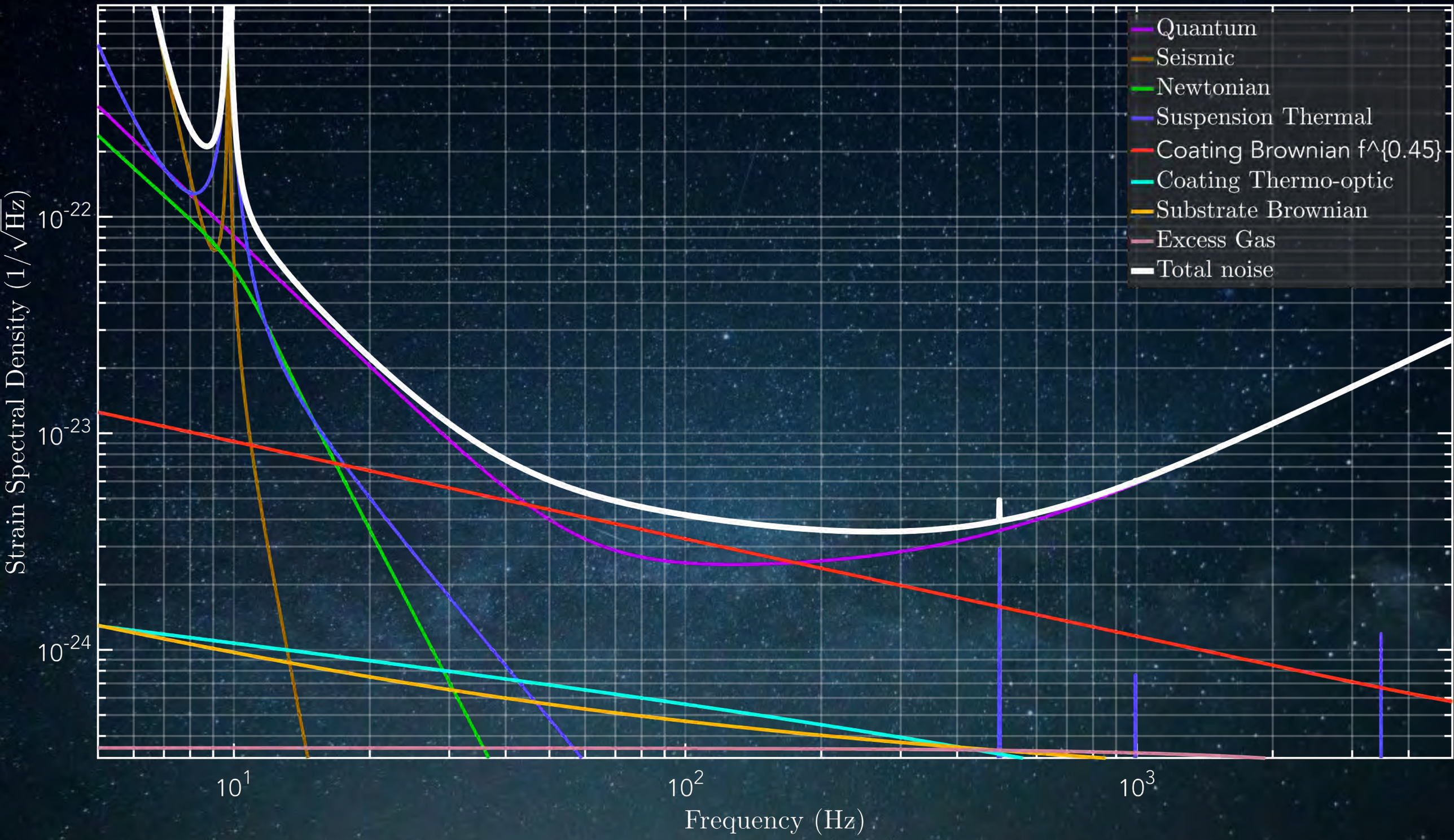
Isolation on ground



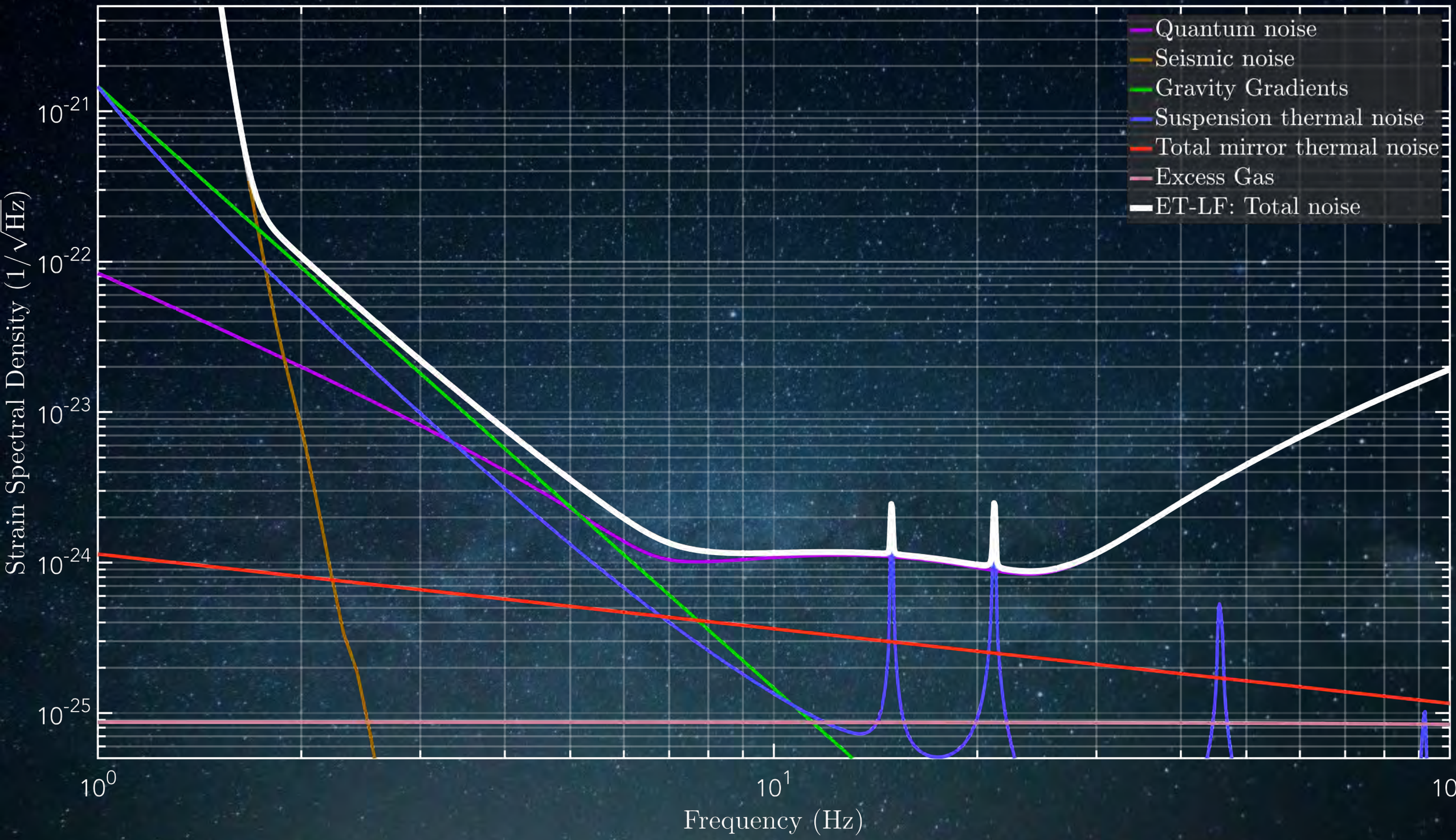


Brett Shapiro, G1400964-v3

aLIGO limiting noise sources (design)



ET low frequency Interferometer



Low frequency disturbances

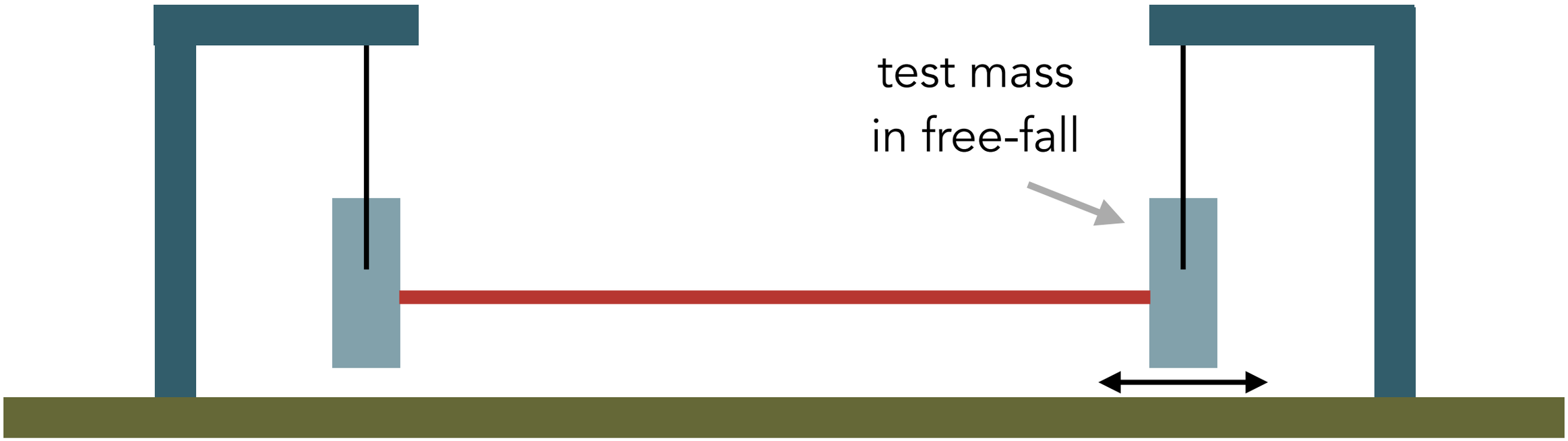
- Going to lower frequency (mHz) is not feasible on ground
  - Go to space!

- What is the measurement principle in space?

- What are the challenges of low-frequency observations in space?

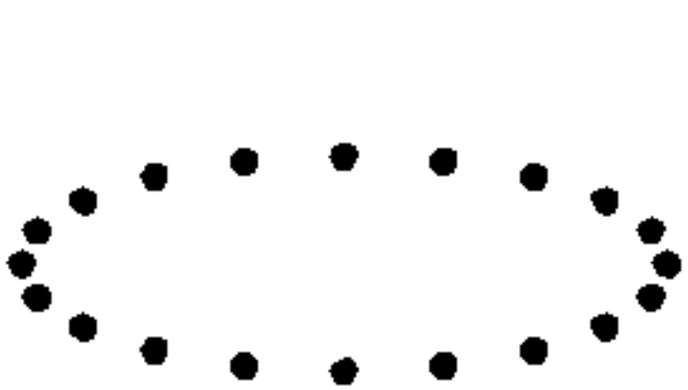
Measurement principle: fundamentals

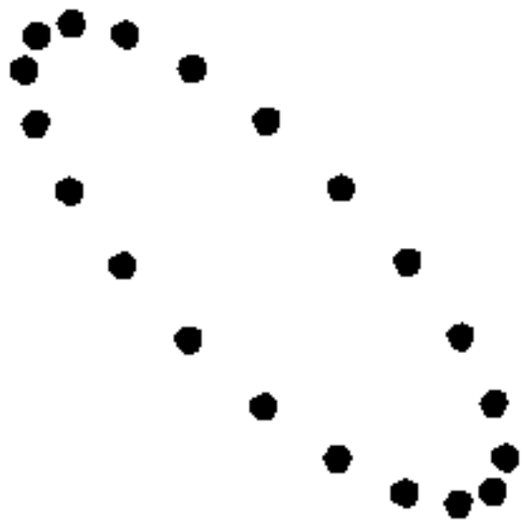
- Aim is to measure the relative motion of free-falling test masses
- On ground test masses are suspended, but are ‘free-falling’ along the measurement axis and above the suspension resonance



Measurement principle: fundamentals

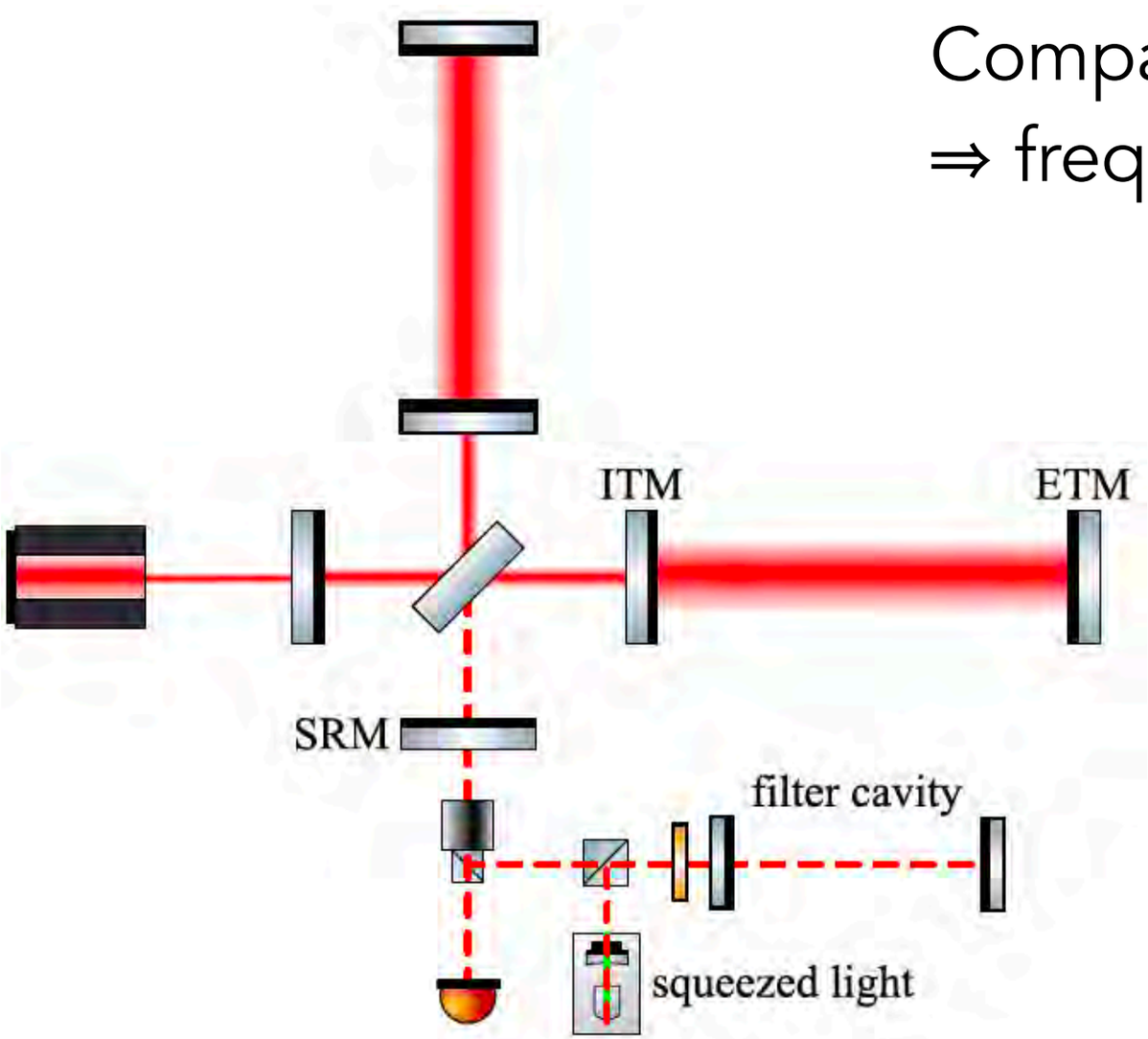
- Compare different pairs of test masses arranged ‘orthogonally’ to combat technical (frequency) noise





Ground based interferometers

Laser wavelength (frequency) is our ‘ruler’
⇒ frequency noise is indistinguishable from GW signal



Compare length changes of each arm
⇒ frequency noise is common to both

Actually use ~equal arm-length interferometer, ~insensitive to frequency noise

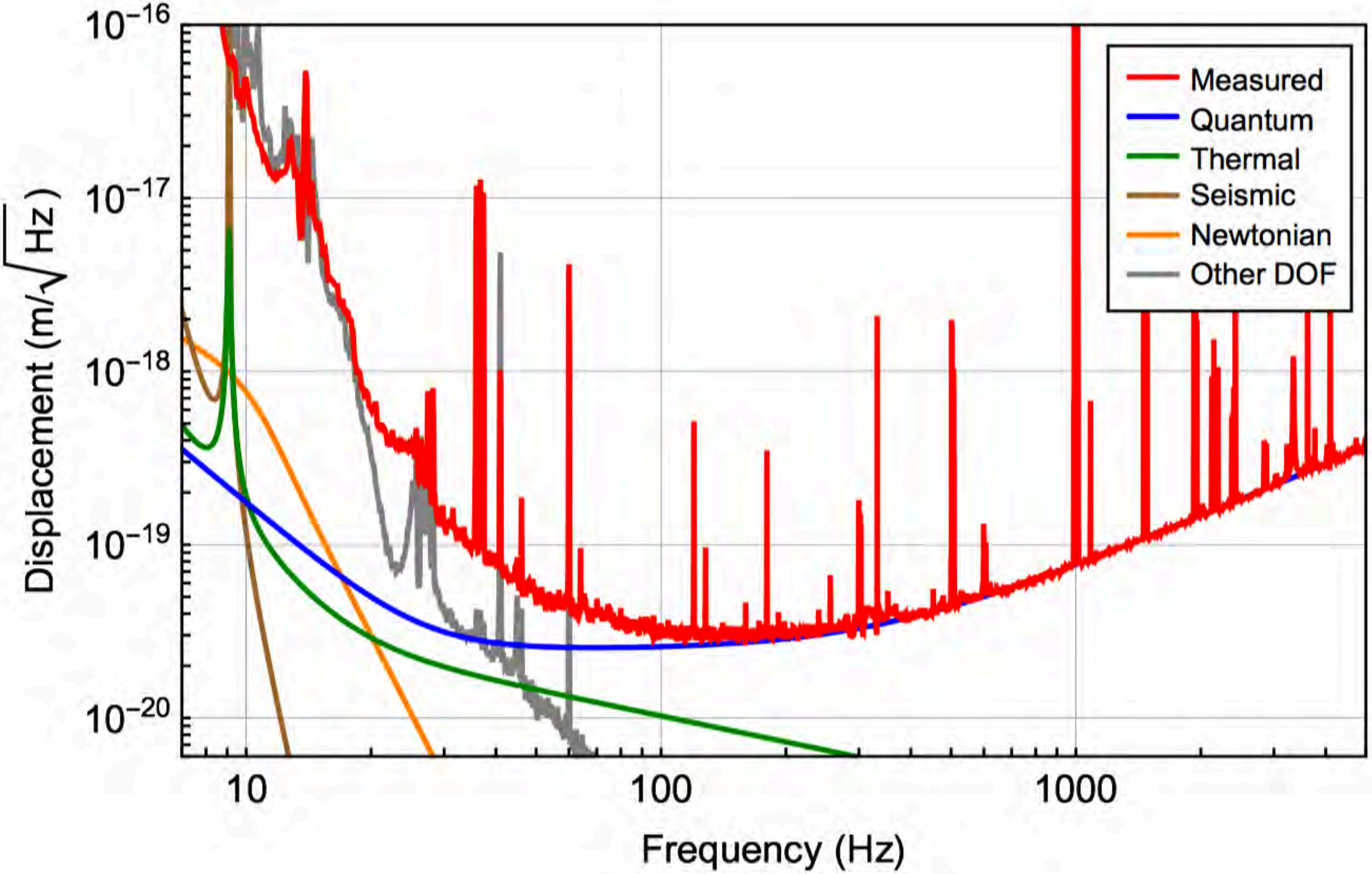
Length stability

- Advanced suspensions result in very quiet mirrors

- Advanced optics and readout result in displacement sensitivity < 10^{-19} m/√Hz at 100Hz!
  - Achievable because mirrors are held at operating point

- Actuation noise is all at lower frequency

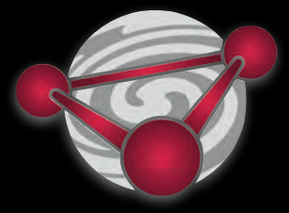
aLIGO displacement sensitivity



Homodyne versus heterodyne

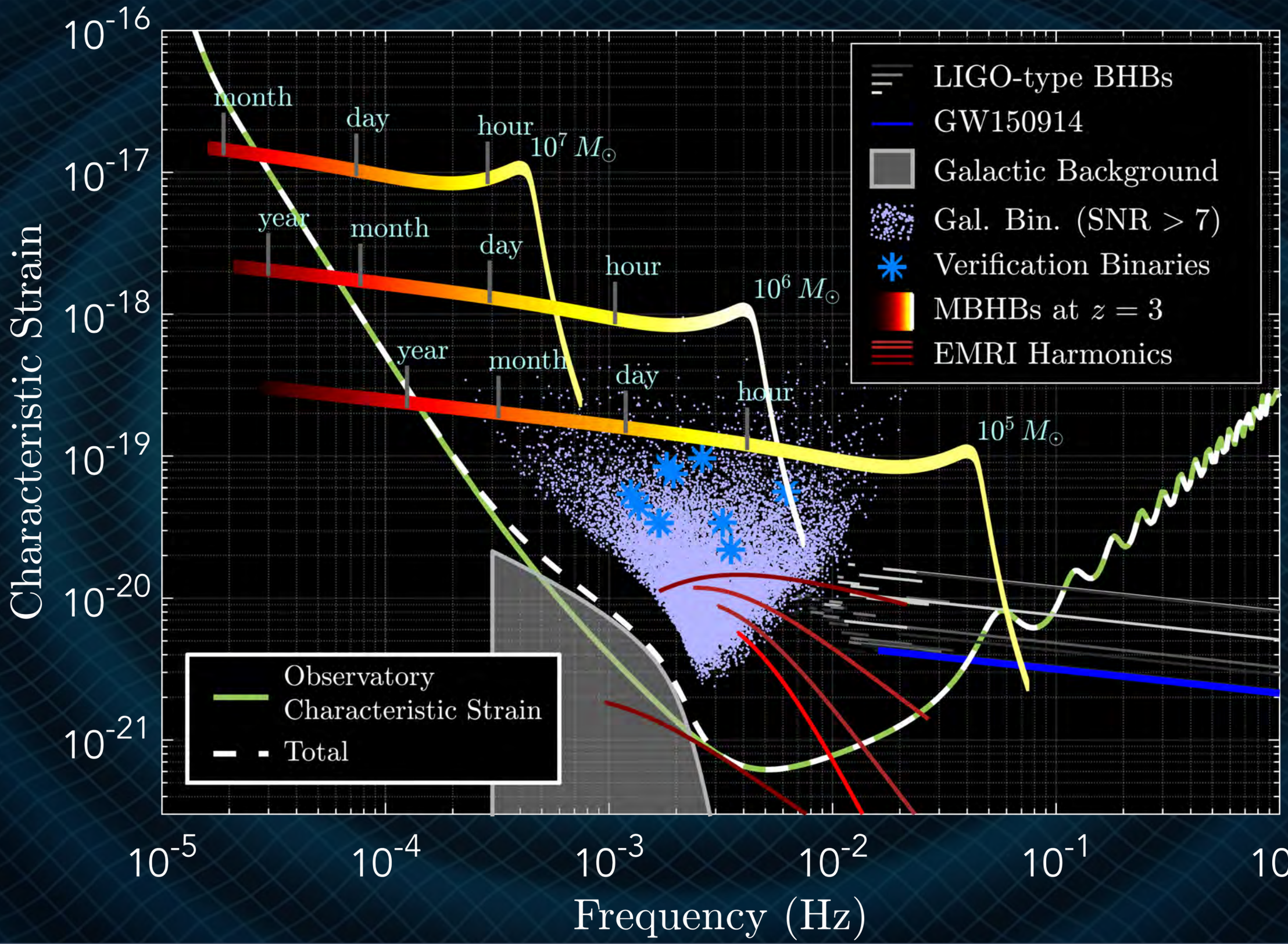
- why can’t we lock on a fringe in space?
- orbital motion of free-falling bodies produces large relative motions
  - would need very strong (and noisy!) feedback forces to maintain ‘fixed' arm-lengths
  - this would need to be a mix of force on S/C (thrusters) and force on test masses
- Use another concept instead: heterodyne interferometry with phase tracking
  - can measure over many fringes

The LISA Concept



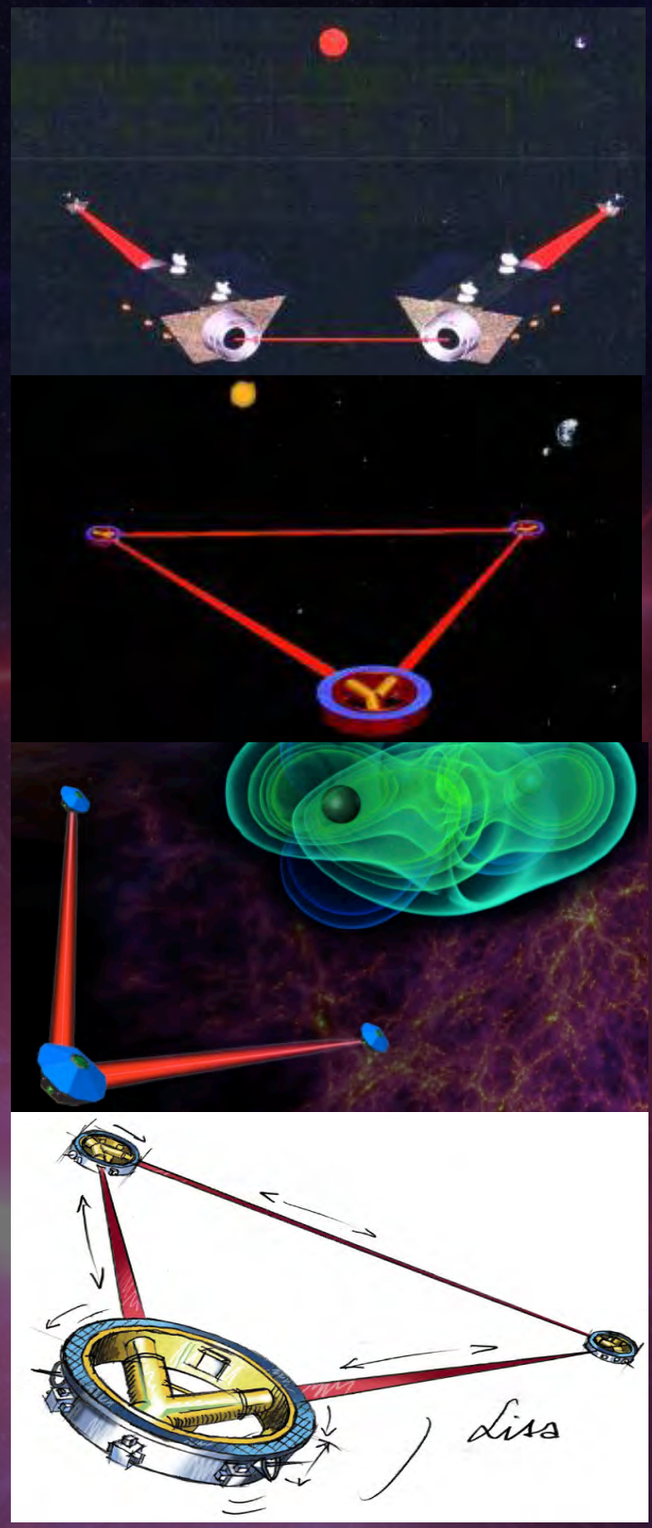
- Probe the change in proper time between pairs of **free-falling** test masses caused by GWs
- Free-falling test masses are housed in **drag-free** spacecraft separated by a few million km
- Proper time is inferred by the time of flight of photons exchanged between the satellites
- We have **multiple transponder links** from which we can form Michelson-like signals

LISA Science



LISA is a mature concept

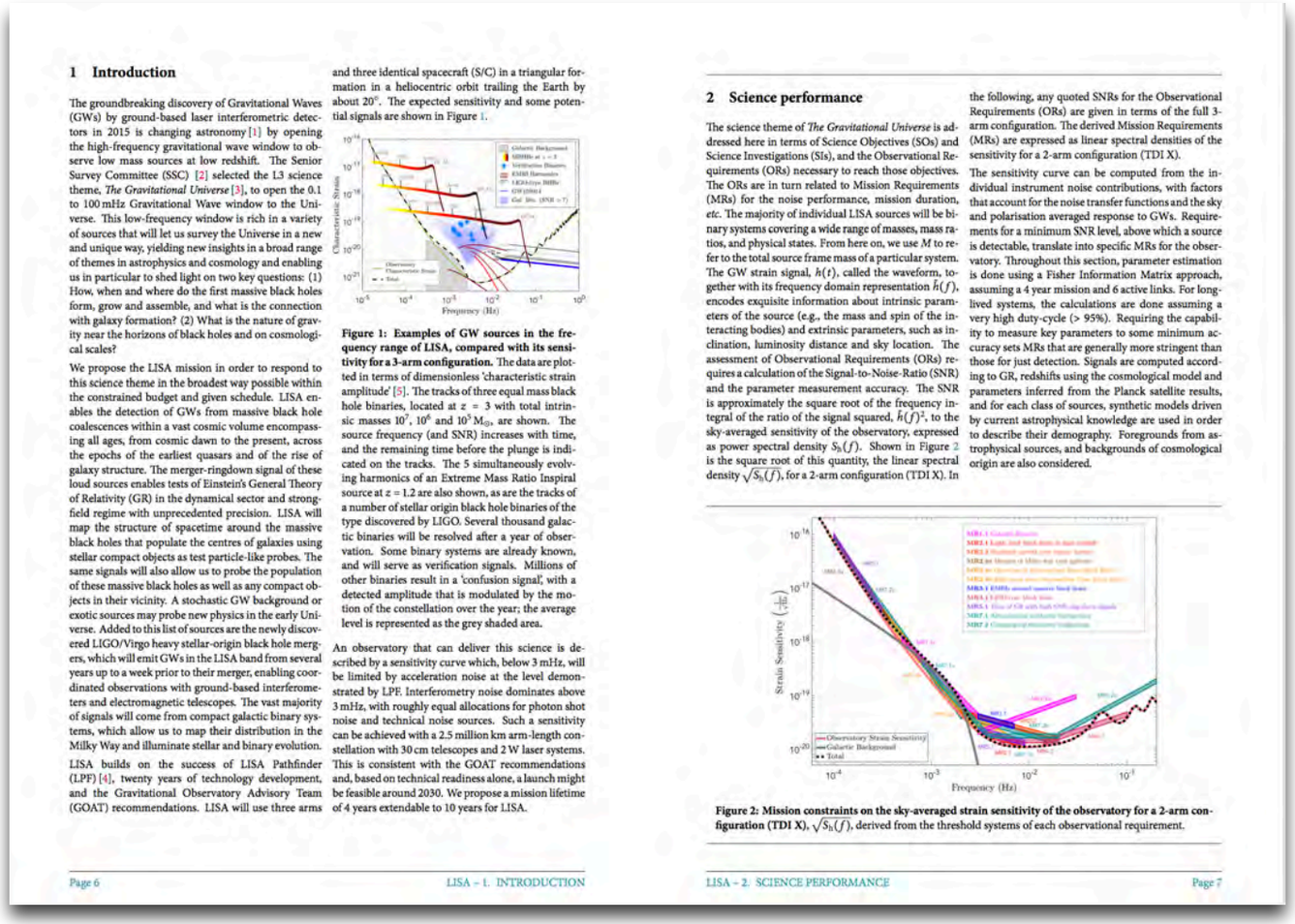
- **1993** M3 proposal for 4 S/C ESA/NASA collaborative mission
- **1995** LISA selected as ESA Cornerstone
- **1997** 3 S/C ESA/NASA LISA appears
- **2005-2011** Joint ESA-NASA Mission Formulation study
- **2012-13** Mission reformulated as ESA-led eLISA (evolving LISA)
- **2013** Gravitational Universe selected as L3 science theme
- **2017** LISA proposal selected as ESA's L3 mission



LISA Proposal for ESA's L3 mission slot

www.lisamission.org

https://arxiv.org/abs/1702.00786



1 Introduction

The groundbreaking discovery of Gravitational Waves (GWs) by ground-based laser interferometric detectors in 2015 is changing astronomy [1] by opening the high-frequency gravitational wave window to observe low mass sources at low redshift. The Senior Survey Committee (SSC) [2] selected the L3 science theme, *The Gravitational Universe* [3], to open the 0.1 to 100 mHz Gravitational Wave window to the Universe. This low-frequency window is rich in a variety of sources that will let us survey the Universe in a new and unique way, yielding new insights in a broad range of themes in astrophysics and cosmology and enabling us in particular to shed light on two key questions: (1) How, when and where do the first massive black holes form, grow and assemble, and what is the connection with galaxy formation? (2) What is the nature of gravity near the horizons of black holes and on cosmological scales?

We propose the LISA mission in order to respond to this science theme in the broadest way possible within the constrained budget and given schedule. LISA enables the detection of GWs from massive black hole coalescences within a vast cosmic volume encompassing all ages, from cosmic dawn to the present, across the epochs of the earliest quasars and of the rise of galaxy structure. The merger-ringdown signal of these loud sources enables tests of Einstein's General Theory of Relativity (GR) in the dynamical sector and strong-field regime with unprecedented precision. LISA will map the structure of spacetime around the massive black holes that populate the centres of galaxies using stellar compact objects as test particle-like probes. The same signals will also allow us to probe the population of these massive black holes as well as any compact objects in their vicinity. A stochastic GW background or exotic sources may probe new physics in the early Universe. Added to this list of sources are the newly discovered LIGO/Virgo heavy stellar-origin black hole mergers, which will emit GWs in the LISA band from several years up to a week prior to their merger, enabling coordinated observations with ground-based interferometers and electromagnetic telescopes. The vast majority of signals will come from compact galactic binary systems, which allow us to map their distribution in the Milky Way and illuminate stellar and binary evolution.

LISA builds on the success of LISA Pathfinder (LPF) [4], twenty years of technology development, and the Gravitational Observatory Advisory Team (GOAT) recommendations. LISA will use three arms and three identical spacecraft (S/C) in a triangular formation in a heliocentric orbit trailing the Earth by about 20°. The expected sensitivity and some potential signals are shown in Figure 1.

Figure 1: Examples of GW sources in the frequency range of LISA, compared with its sensitivity for a 3-arm configuration. The data are plotted in terms of dimensionless 'characteristic strain amplitude' [5]. The tracks of three equal mass black hole binaries, located at $z = 3$ with total intrinsic masses 10^7, 10^6 and $10^5\,M_\odot$, are shown. The source frequency (and SNR) increases with time, and the remaining time before the plunge is indicated on the tracks. The 5 simultaneously evolving harmonics of an Extreme Mass Ratio Inspiral source at $z = 1.2$ are also shown, as are the tracks of a number of stellar origin black hole binaries of the type discovered by LIGO. Several thousand galactic binaries will be resolved after a year of observation. Some binary systems are already known, and will serve as verification signals. Millions of other binaries result in a 'confusion signal', with a detected amplitude that is modulated by the motion of the constellation over the year; the average level is represented as the grey shaded area.

An observatory that can deliver this science is described by a sensitivity curve which, below 3 mHz, will be limited by acceleration noise at the level demonstrated by LPF. Interferometry noise dominates above 3 mHz, with roughly equal allocations for photon shot noise and technical noise sources. Such a sensitivity can be achieved with a 2.5 million km arm-length constellation with 30 cm telescopes and 2 W laser systems. This is consistent with the GOAT recommendations and, based on technical readiness alone, a launch might be feasible around 2030. We propose a mission lifetime of 4 years extendable to 10 years for LISA.

2 Science performance

The science theme of *The Gravitational Universe* is addressed here in terms of Science Objectives (SOs) and Science Investigations (SIs), and the Observational Requirements (ORs) necessary to reach those objectives. The ORs are in turn related to Mission Requirements (MRs) for the noise performance, mission duration, *etc*. The majority of individual LISA sources will be binary systems covering a wide range of masses, mass ratios, and physical states. From here on, we use M to refer to the total source frame mass of a particular system. The GW strain signal, $h(t)$, called the waveform, together with its frequency domain representation $\tilde{h}(f)$, encodes exquisite information about intrinsic parameters of the source (e.g., the mass and spin of the interacting bodies) and extrinsic parameters, such as inclination, luminosity distance and sky location. The assessment of Observational Requirements (ORs) requires a calculation of the Signal-to-Noise-Ratio (SNR) and the parameter measurement accuracy. The SNR is approximately the square root of the frequency integral of the ratio of the signal squared, $\tilde{h}(f)^2$, to the sky-averaged sensitivity of the observatory, expressed as power spectral density $S_h(f)$. Shown in Figure 2 is the square root of this quantity, the linear spectral density $\sqrt{S_h(f)}$, for a 2-arm configuration (TDI X). In the following, any quoted SNRs for the Observational Requirements (ORs) are given in terms of the full 3-arm configuration. The derived Mission Requirements (MRs) are expressed as linear spectral densities of the sensitivity for a 2-arm configuration (TDI X).

The sensitivity curve can be computed from the individual instrument noise contributions, with factors that account for the noise transfer functions and the sky and polarisation averaged response to GWs. Requirements for a minimum SNR level, above which a source is detectable, translate into specific MRs for the observatory. Throughout this section, parameter estimation is done using a Fisher Information Matrix approach, assuming a 4 year mission and 6 active links. For long-lived systems, the calculations are done assuming a very high duty-cycle (> 95%). Requiring the capability to measure key parameters to some minimum accuracy sets MRs that are generally more stringent than those for just detection. Signals are computed according to GR, redshifts using the cosmological model and parameters inferred from the Planck satellite results, and for each class of sources, synthetic models driven by current astrophysical knowledge are used in order to describe their demography. Foregrounds from astrophysical sources, and backgrounds of cosmological origin are also considered.

Figure 2: Mission constraints on the sky-averaged strain sensitivity of the observatory for a 2-arm configuration (TDI X), $\sqrt{S_h(f)}$, derived from the threshold systems of each observational requirement.

20th June 2017

LISA selected as ESA's L3 mission!

“The LISA trio of satellites to detect gravitational waves from space has been selected as the third large-class mission in ESA’s Science programme, while the Plato exoplanet hunter moves into development.

These important milestones were decided upon during a meeting of ESA’s Science Programme Committee today, and ensure the continuation of ESA’s Cosmic Vision plan through the next two decades.”

http://www.esa.int/Our_Activities/Space_Science/

?

LISA Measurement concept: Challenges

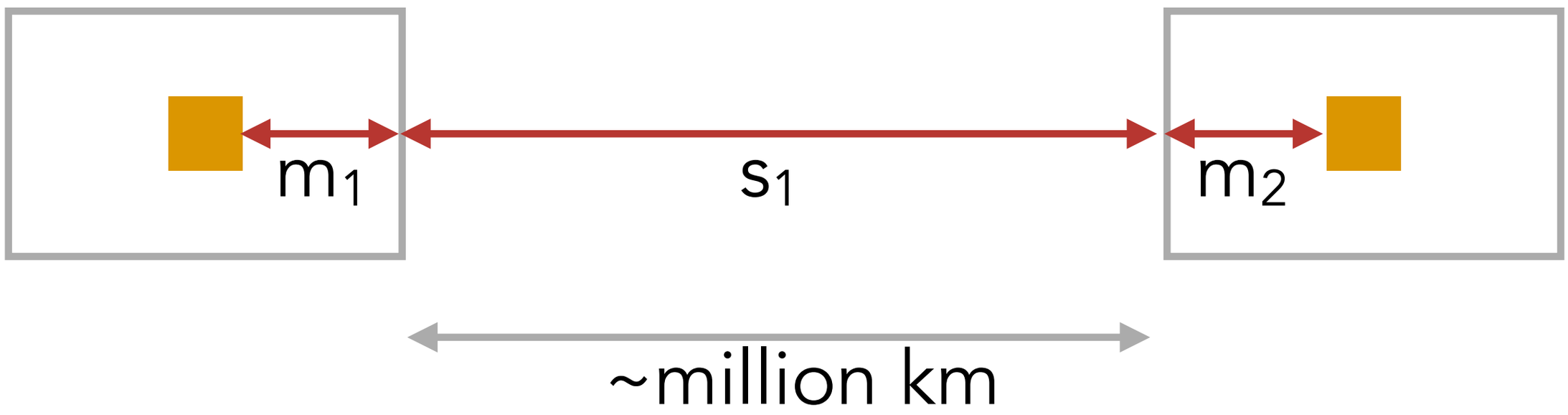
- protect test masses from external forces which move test masses away from perfect geodesics
  - e.g. solar wind
- can’t do interferometry as on ground (locked cavities)
  - high dynamics
  - beam divergence means most of the light is lost
  - need a transponder scheme

Test mass free-fall in space

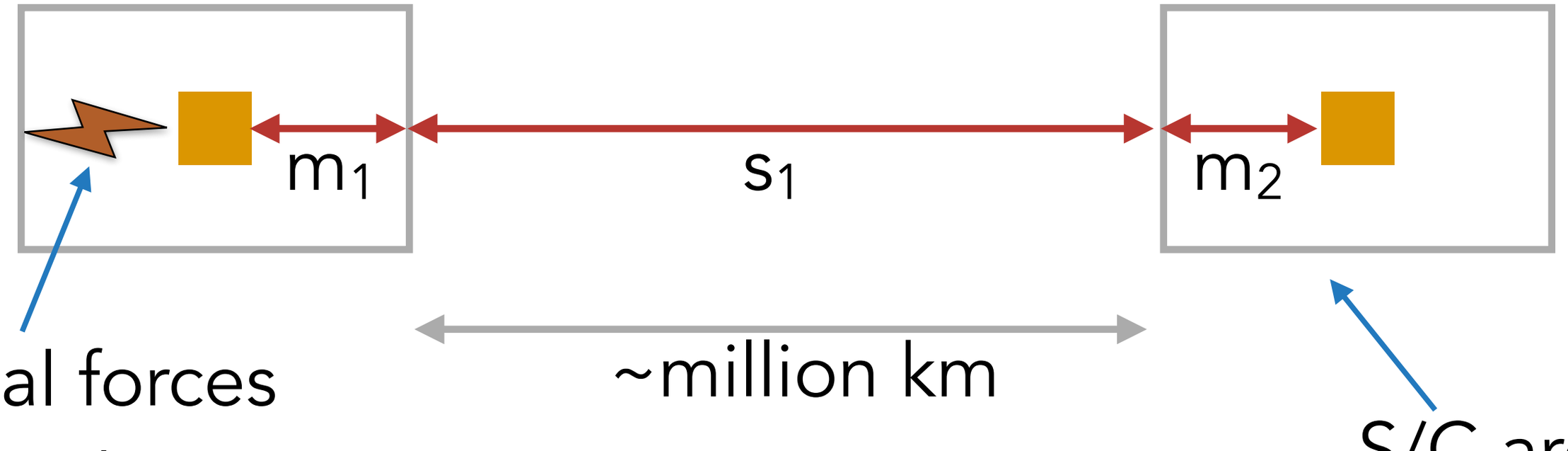
- Shield free-falling test mass (TM) in a spacecraft (S/C)
- Need to ensure local forces between S/C and TM are ‘low enough’
  - less relevant at higher frequencies on ground as they fall off with $1/f^2$
  - S/C has no mechanical connection to the TM
    - drag-free S/C control
    - issues of charge accumulation on the TM (see later)

Measurement principle

- make 3 measurements:
  - TM to S/C
  - S/C to S/C (contains the GW signal!)
  - S/C to TM
- Combine to get TM to TM



What can possibly go wrong?



local forces

disturb TM

magnetic force
thermal
residual gas
force gradients

S/C are ‘shaking’

thrusters for drag-free control
external forces (e.g. solar wind)

Lasers over millions of km

beam waist diameter

- laser beam with gaussian profile diverges as

$$w(x) = w_0 \sqrt{1 + \left(\frac{z}{z_{\mathrm{R}}}\right)^2}$$

beam width

Rayleigh range

- where the Rayleigh range is given by

$$z_{\mathrm{R}} = \frac{\pi w_0^2}{\lambda}$$

wavelength

- Far from the beam waist (large z), we have $w(x) \approx \frac{z\lambda}{\pi w_0}$

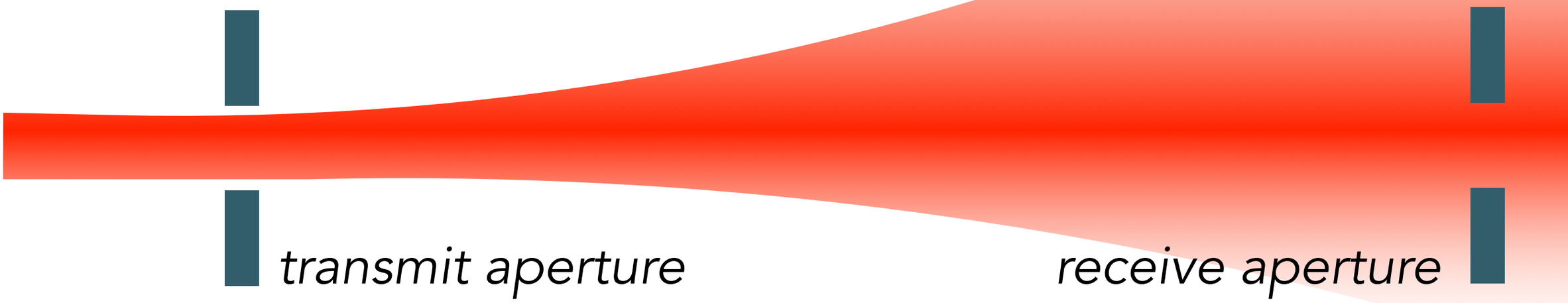
- For 2.5 million km, and laser wavelength of 1064 nm

$$w(x) = \frac{2.5 \times 10^9 [\mathrm{m}] \times 1064 \times 10^{-9} [\mathrm{m}]}{\pi \; 13.4 [\mathrm{cm}]} \approx 6.4\,\mathrm{km}$$

w(x) defined where amplitude falls to 1/e ~ 0.37

Lasers over millions of km

- if we send out a 2W beam from S/C 1, how much power do we intercept at S/C 2?
  - depends on:
    - distance, receiving aperture (telescope), etc
- calculations on previous page assume we can get all the light out of the transmitting telescope
  - in practice, not practical, requires too large a telescope
- Rather we should optimise transmit beam properties to maximise intensity over receiving telescope



Optimising beam waist and received power

- According to [1], the intensity of light received on-axis from a focused, apertured gaussian beam can be expressed as

$$I_0 = \frac{2P}{\pi\omega^2}\left[\frac{\left(\exp\left[-(a/\omega)^2\right] - \exp\left[-(b/\omega)^2\right]\right)^2 + 4\exp\left[-(a^2+b^2)/\omega^2\right]\sin^2\left[\pi(b^2-a^2)(z-R)/2\lambda zR\right]}{(\lambda z/\pi\omega^2)^2 + \left[(z-R)/R\right]^2}\right],$$

- Ouch… apply some assumptions:
  - The arm-length is very long compared to any beam properties, so L→∞
  - We set $\omega = \alpha\frac{d_{\rm tel}}{2}$

    where d_{tel} is the diameter of the transmitting telescope, and **α** is the waist radius in units of the telescope radius
  - The outer radius of the annulus, *b*, is set to $d_{tel}/2$
  - The intensity is considered at the far SC, i.e., where *z=L*

[1] Holmes D, Avizonis P and Wrolstad K 1970 Appl. Opt. 9 2179–80

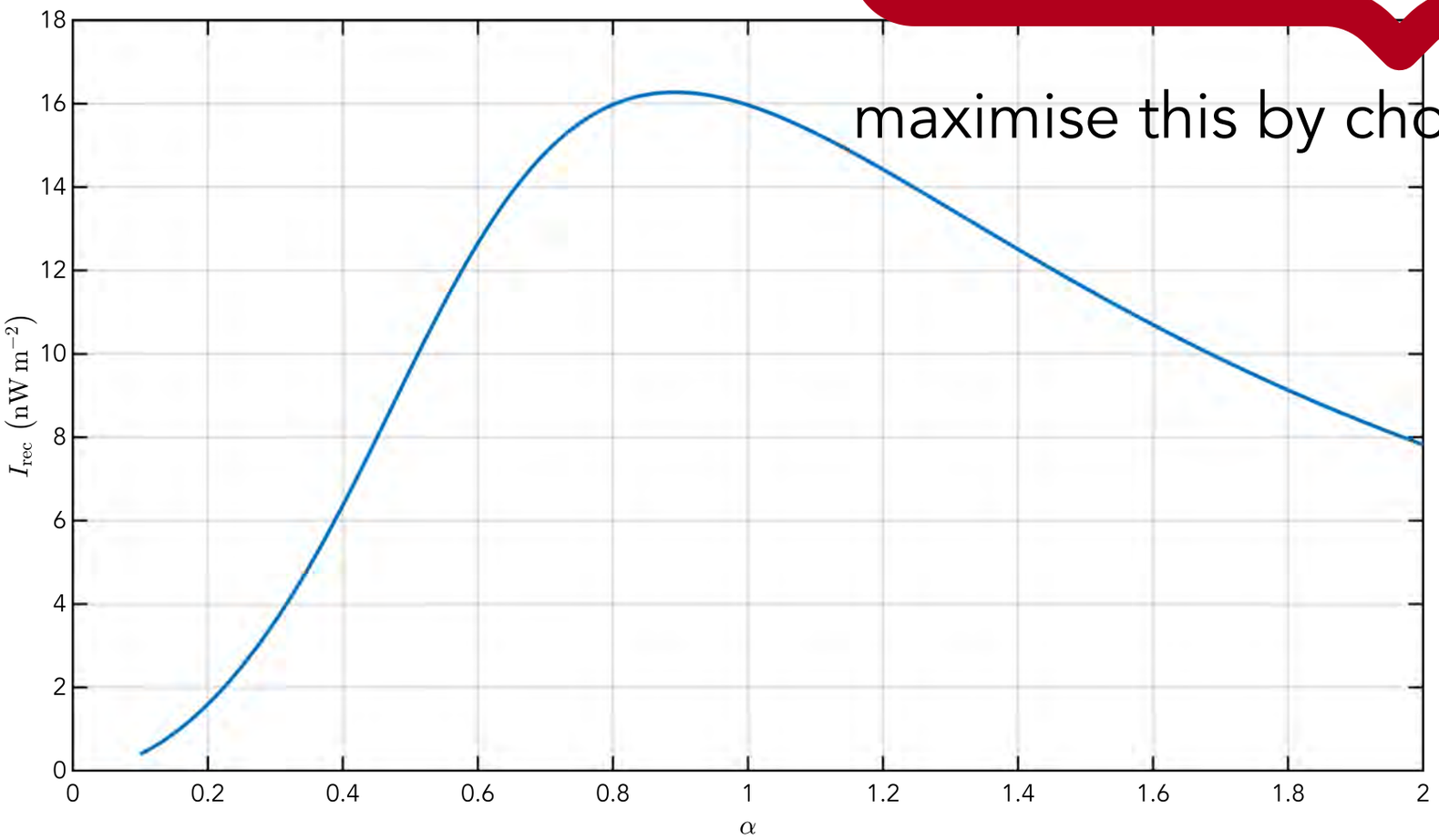
[2] S Barke et al 2015 Class. Quantum Grav. 32 095004

Far-field intensity

- Applying the assumptions gives an expression for the far-field intensity:

$$I_{\mathrm{rec}} = \frac{\pi P_{\mathrm{tel}} d_{\mathrm{tel}}^2}{2L^2\lambda^2} \alpha^2 \exp\left[-2/\alpha^2\right] \left(\exp\left[1/\alpha^2\right] - 1\right)^2 ,$$

maximise this by choosing beam waist



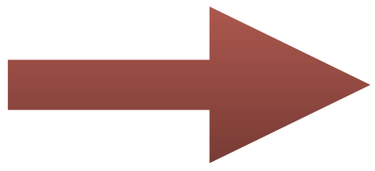
Received power

- Received power depends on
  - far-field intensity
  - receiving aperture (telescope)

$$P_{\rm rec} = \pi \left(\frac{D_{\rm tel}}{2} \right)^2 I_{\rm rec},$$

Some typical values for LISA:

L = 2.5 Gm
Pout = 1.25 W
D_{tel} = 30 cm



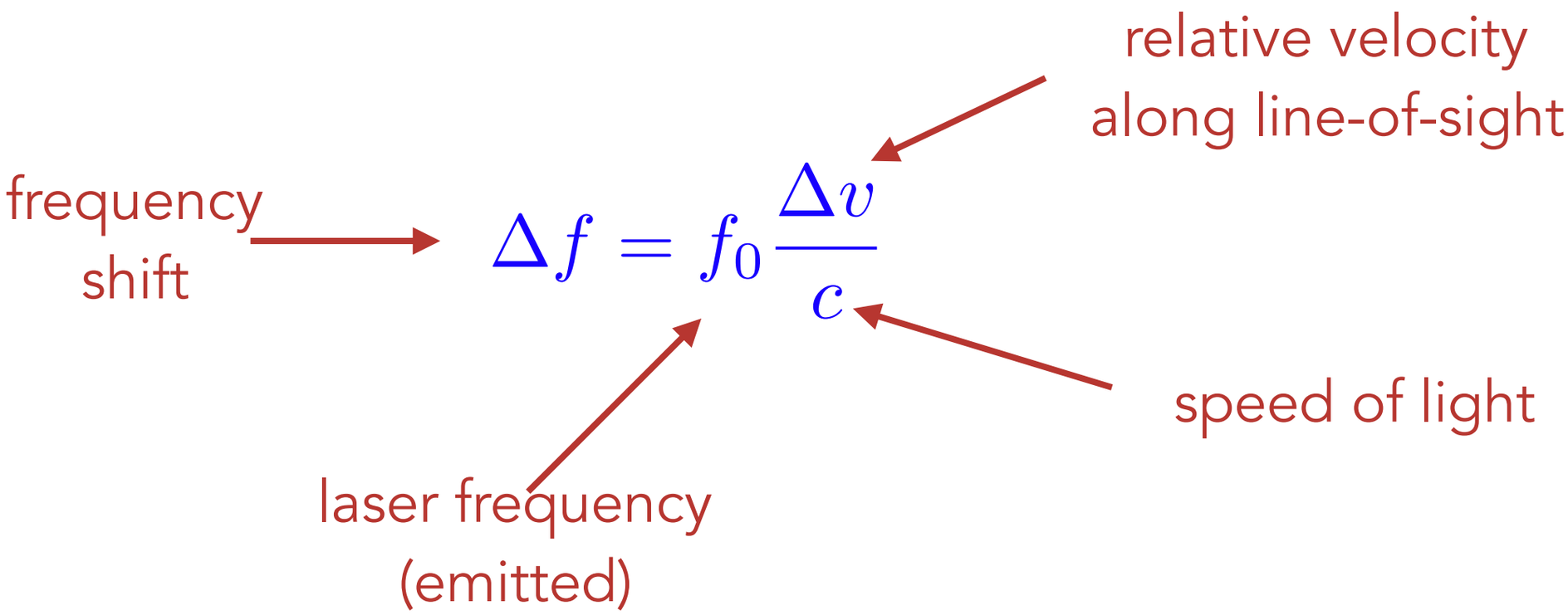
Irec = 10.25 [nW/m^2]
Prec = 507.31 [pW]

So where are we?

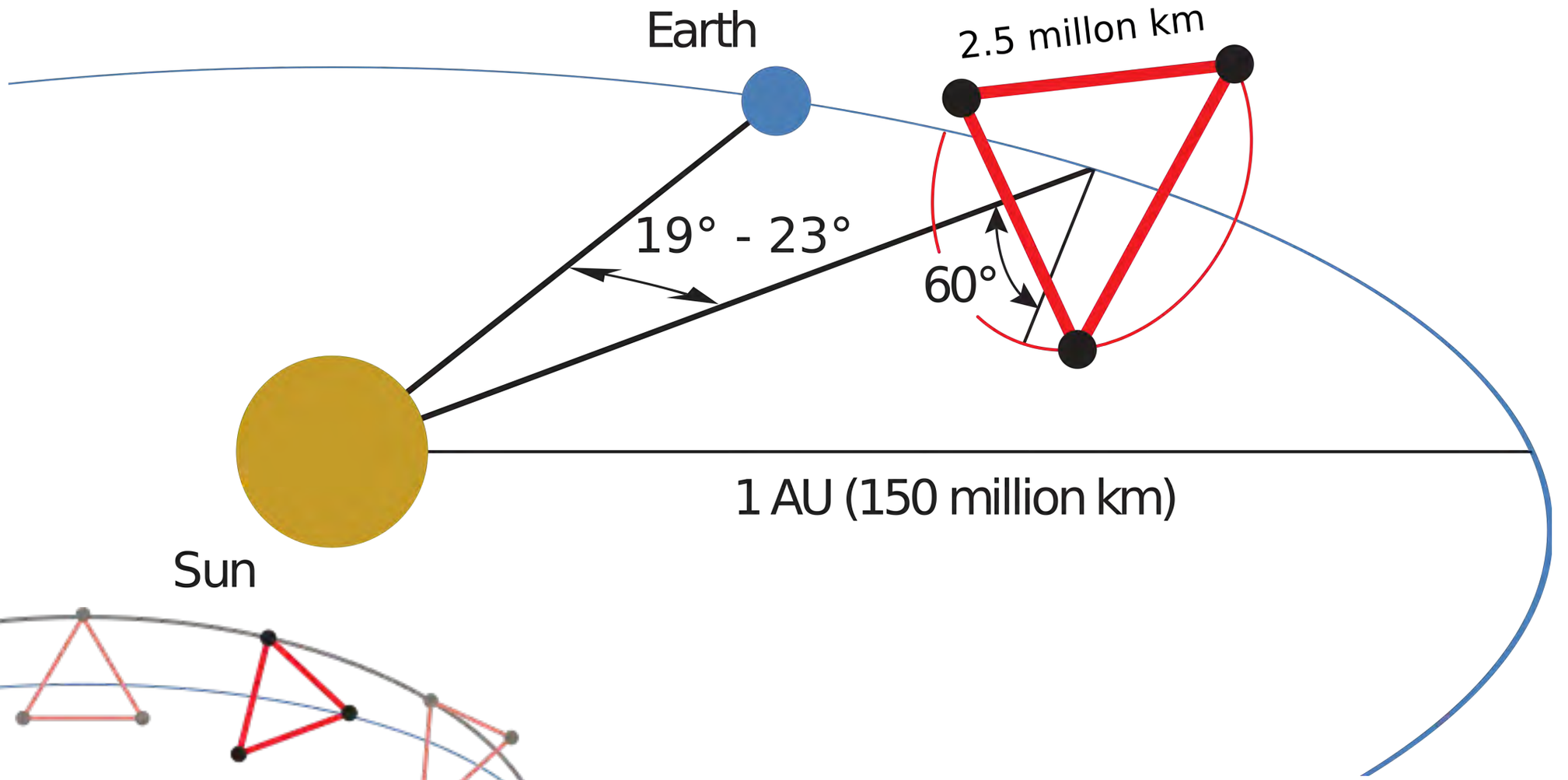
- We use lasers to measure relative motion of free-falling test masses
- But received power is too low to do a round-trip measurement
- Homodyne detection is challenging due to large dynamic range
- Use heterodyne interferometry and track the beats between lasers
  - need to be able to detect these beats!

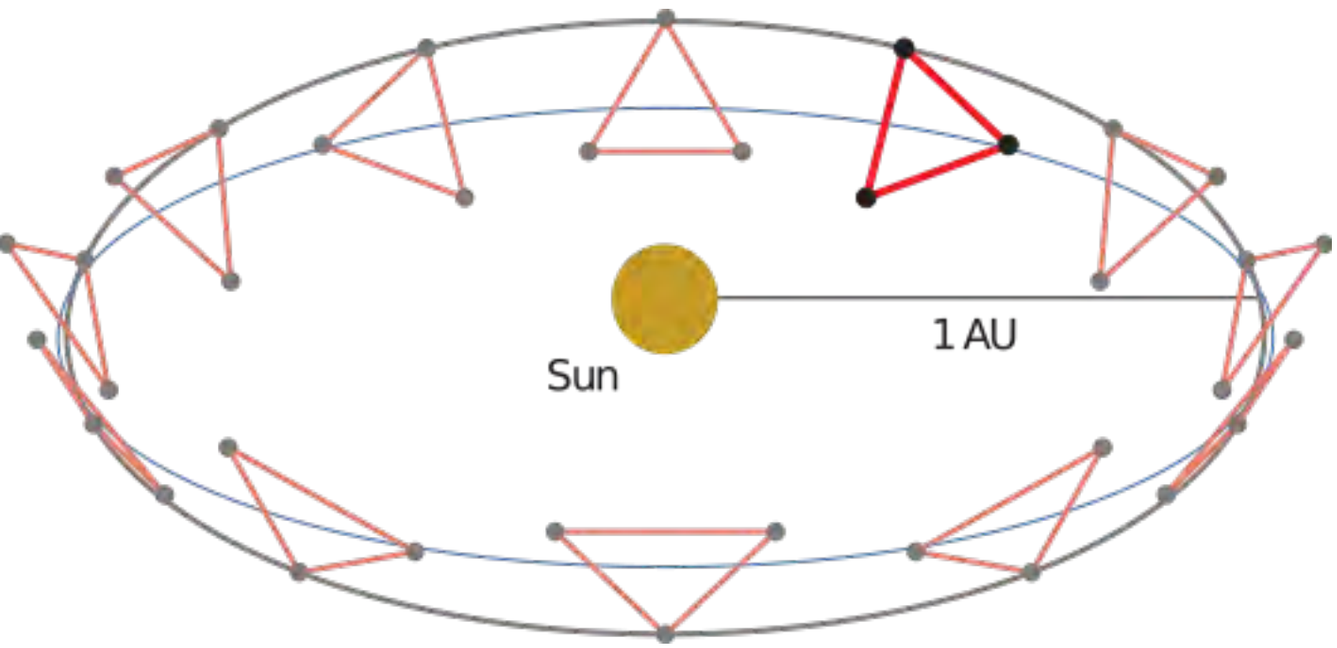
Features of this approach

- Transmitted lasers are subject to doppler shifts due to orbital motions
- Line-of-sight relative velocities of S/C impose frequency shifts on received light

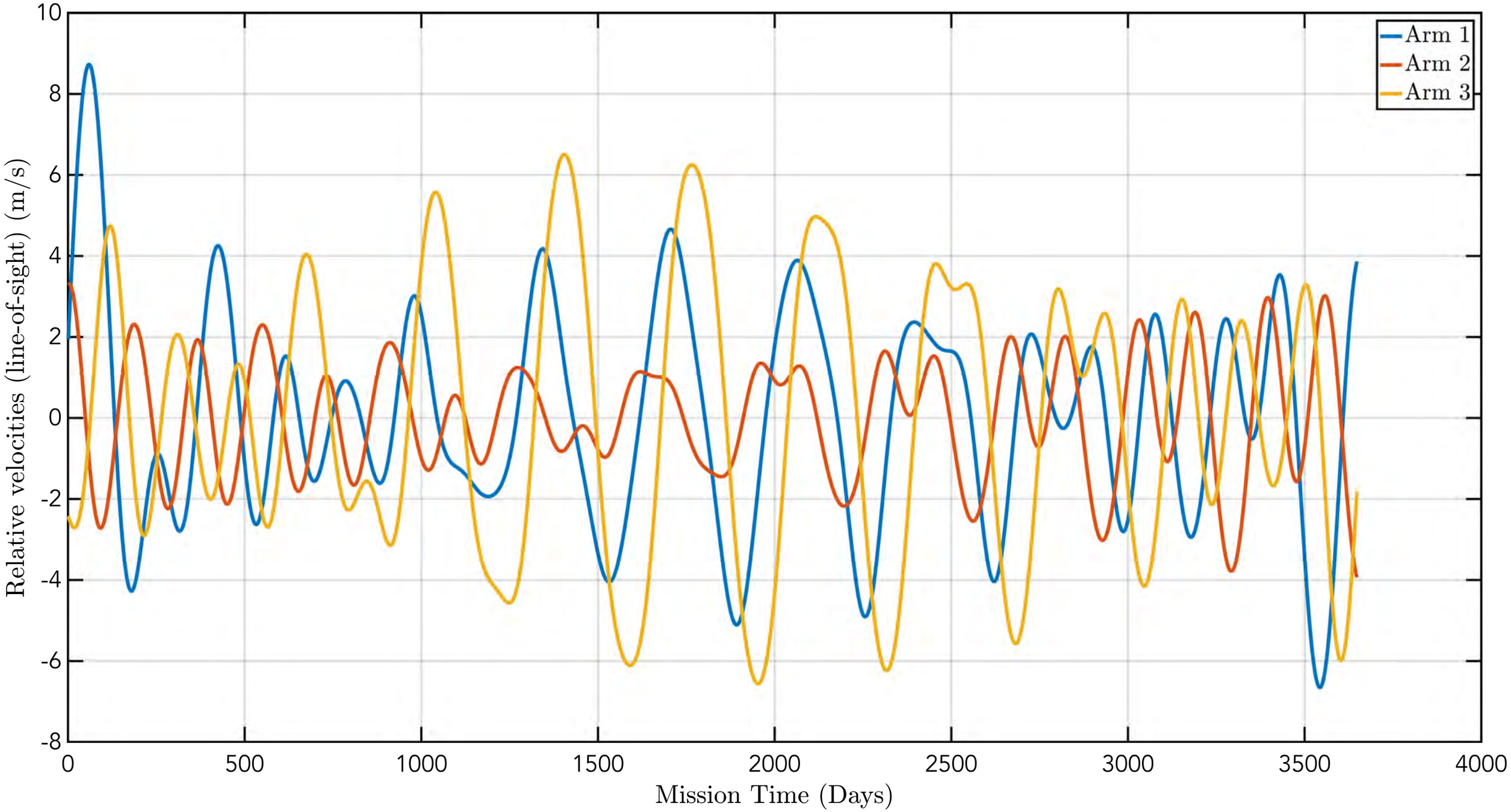


Orbital motions

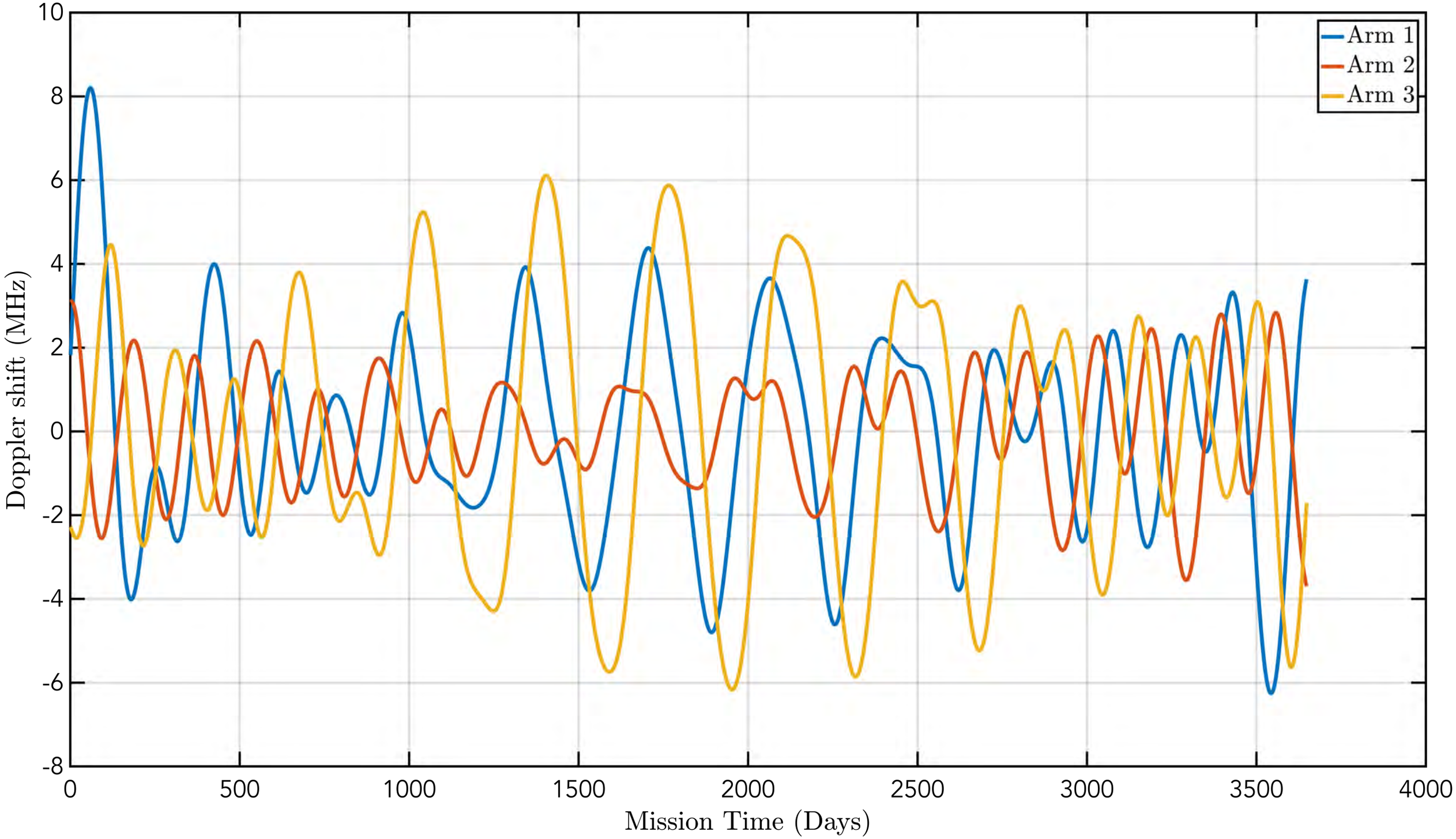




Relative velocities



Doppler Shifts



Beat detection

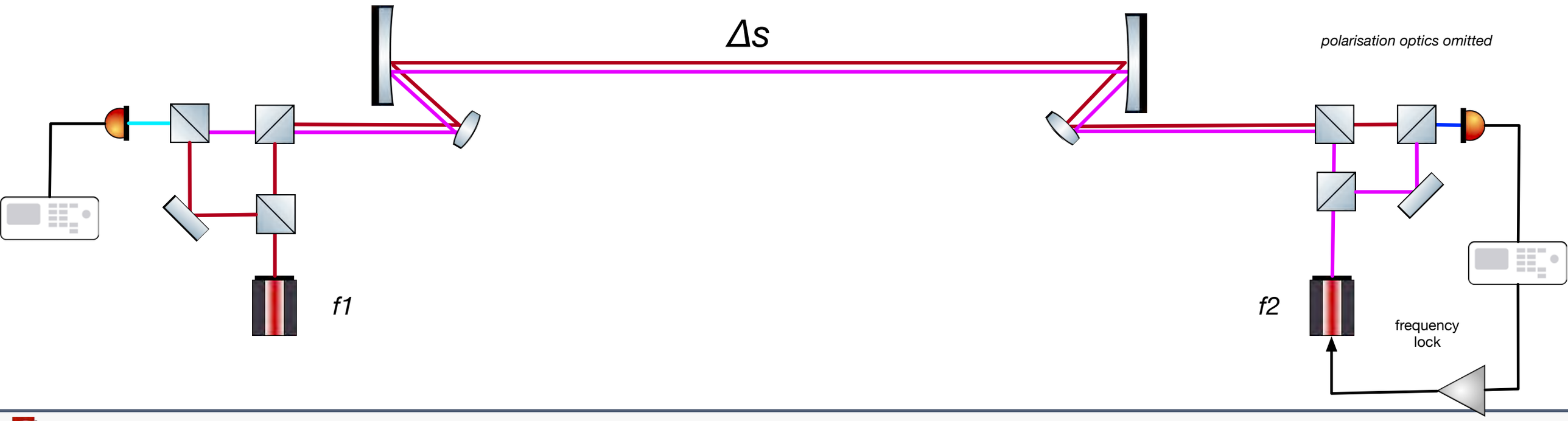
- Need photodiodes which can track beats across many 10's MHz

- Technically very challenging at the noise levels we require

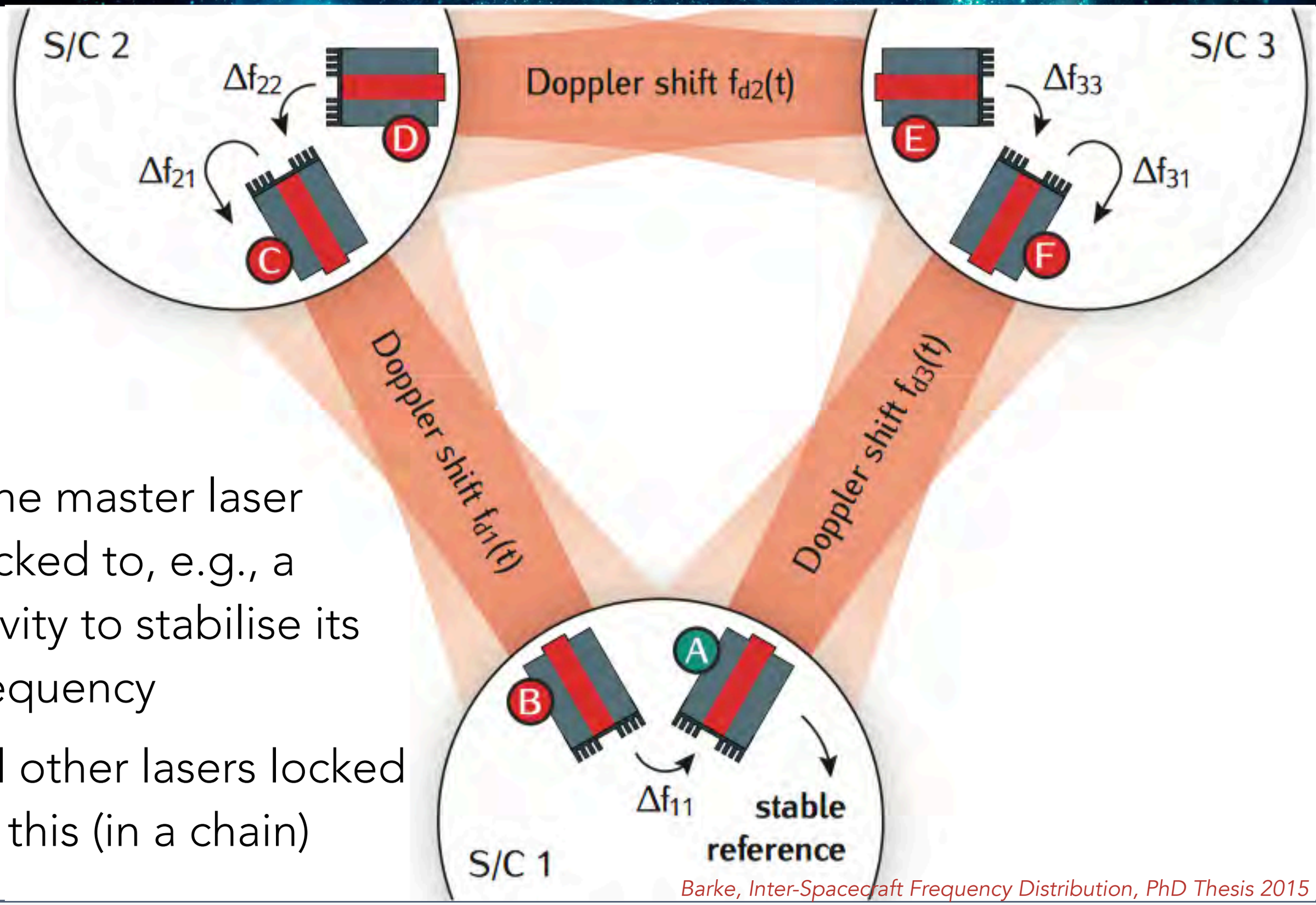
- Need to 'control' the beats to remain in the sensitive region of photodiodes

Transponder-like scheme in LISA

- send out one beam to far S/C
- in far S/C compare received beam to local beam
  - control difference of the two laser frequencies
  - Laser 2 is now locked to Laser 1
    - f2-f1 is kept constant
- Far S/C returns this Laser 2 to S/C 1
  - measure phase difference between two lasers
    - encodes GW signal + S/C motion + frequency noise of Laser 1



Full constellation

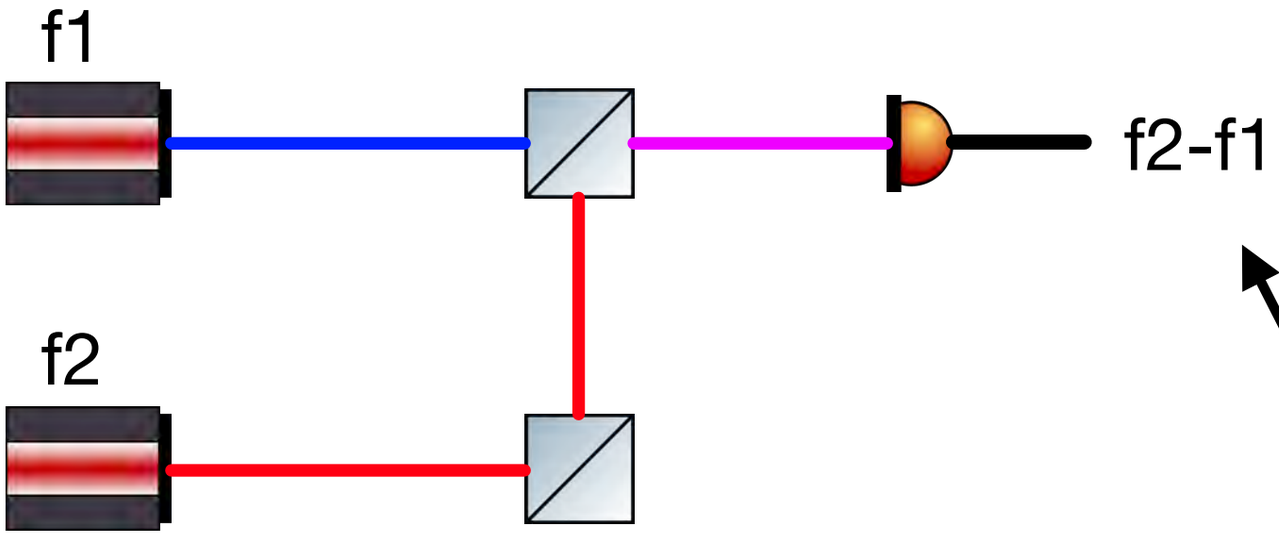


- One master laser locked to, e.g., a cavity to stabilise its frequency
- All other lasers locked to this (in a chain)

Barke, Inter-Spacecraft Frequency Distribution, PhD Thesis 2015

Side bar: Heterodyne interferometry

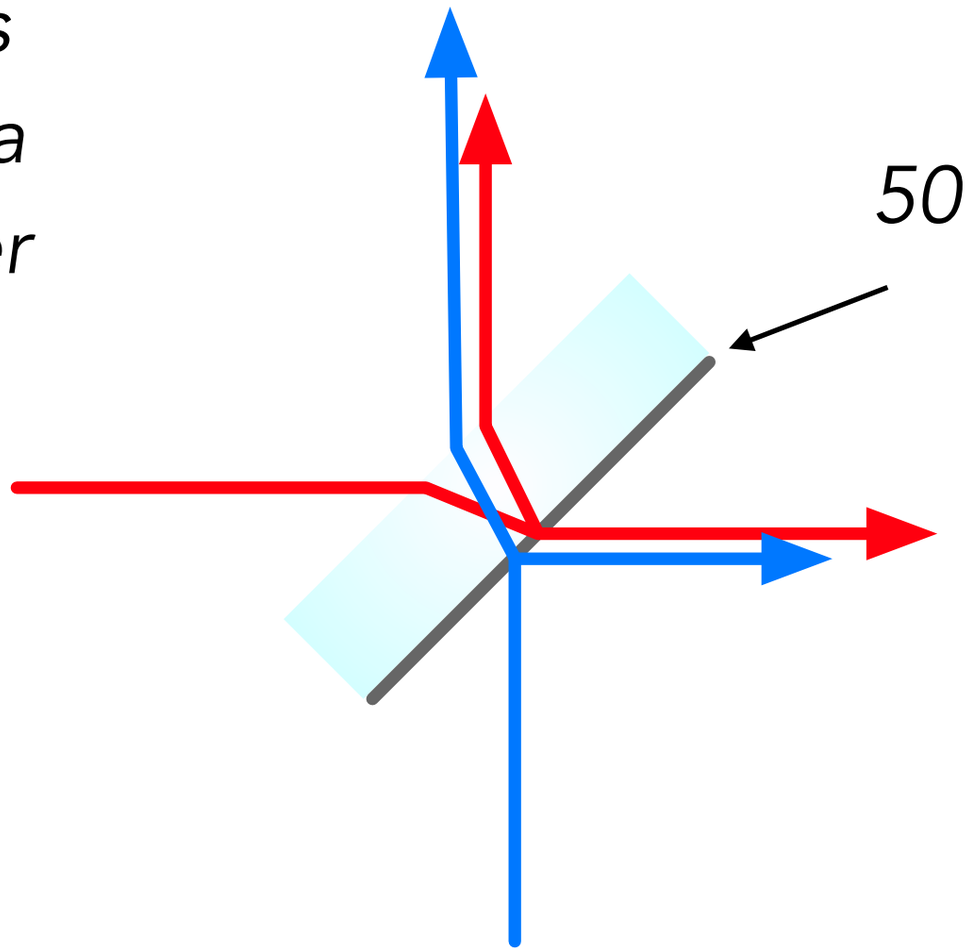
- how does it work?

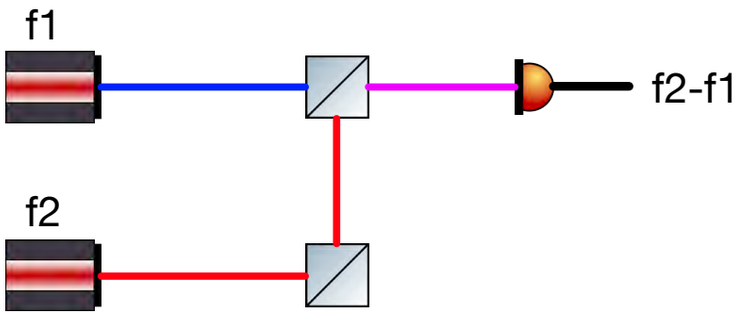


Phase of beat signal depends on path lengths

Interference

2 light fields combine at a beam splitter

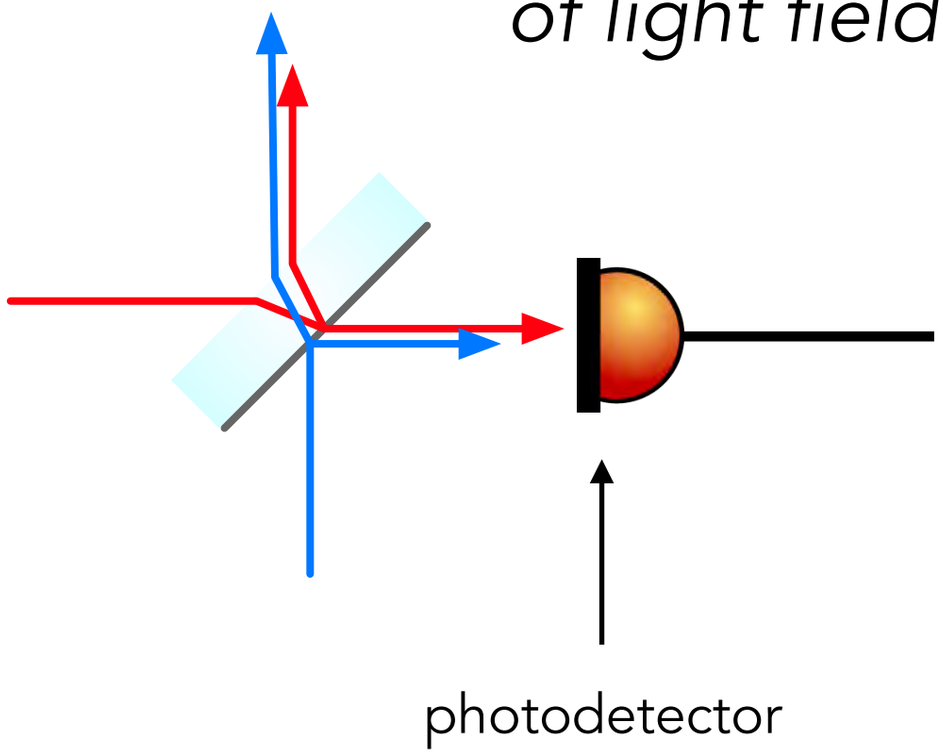




50/50 beam splitter

Detection

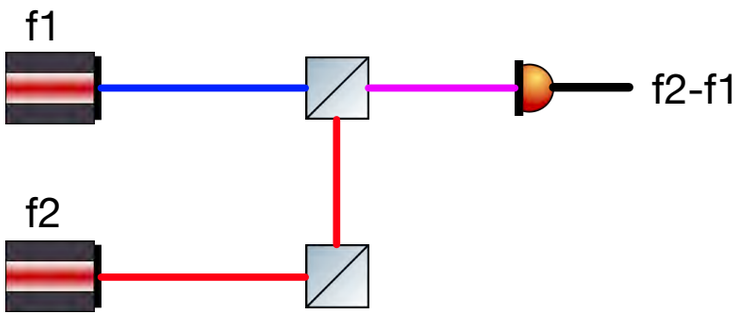
Sum amplitudes of light field!



$$\nu = \frac{c}{\lambda}$$
$$= \frac{299792458\ \left[\mathrm{m\,s^{-1}}\right]}{1064\,\left[\mathrm{nm}\right]}$$
$$= 0.28\,\mathrm{THz}$$

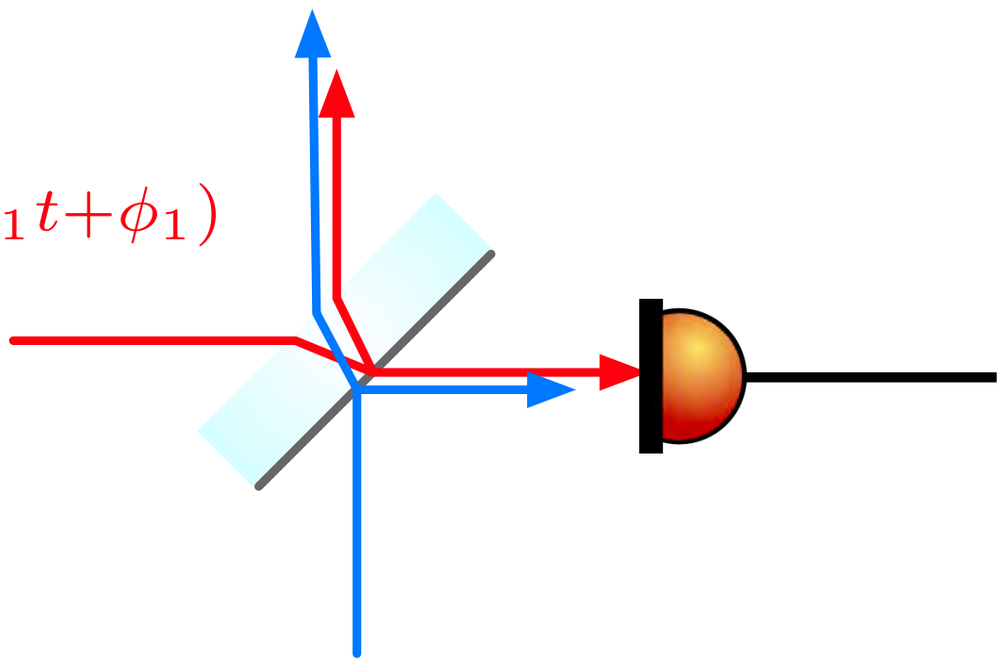
photodetector is a 'square law' detector

Note: carrier frequency is neglected throughout. It is THz, and can't be detected. We only detect average light-fields.



Do the math…

$$s_1 = a_1 e^{(i\omega_1 t + \phi_1)}$$



$$s_2 = a_2 e^{(i\omega_2 t + \phi_2)}$$

$$I_{\rm pd} \propto |s_1 + s_2|^2$$

$$I_{\rm pd} \propto |a_1|^2 + |a_2|^2 + 2|a_1||a_2| \cos\left((\omega_2 - \omega_1)t + \Delta\phi\right)$$

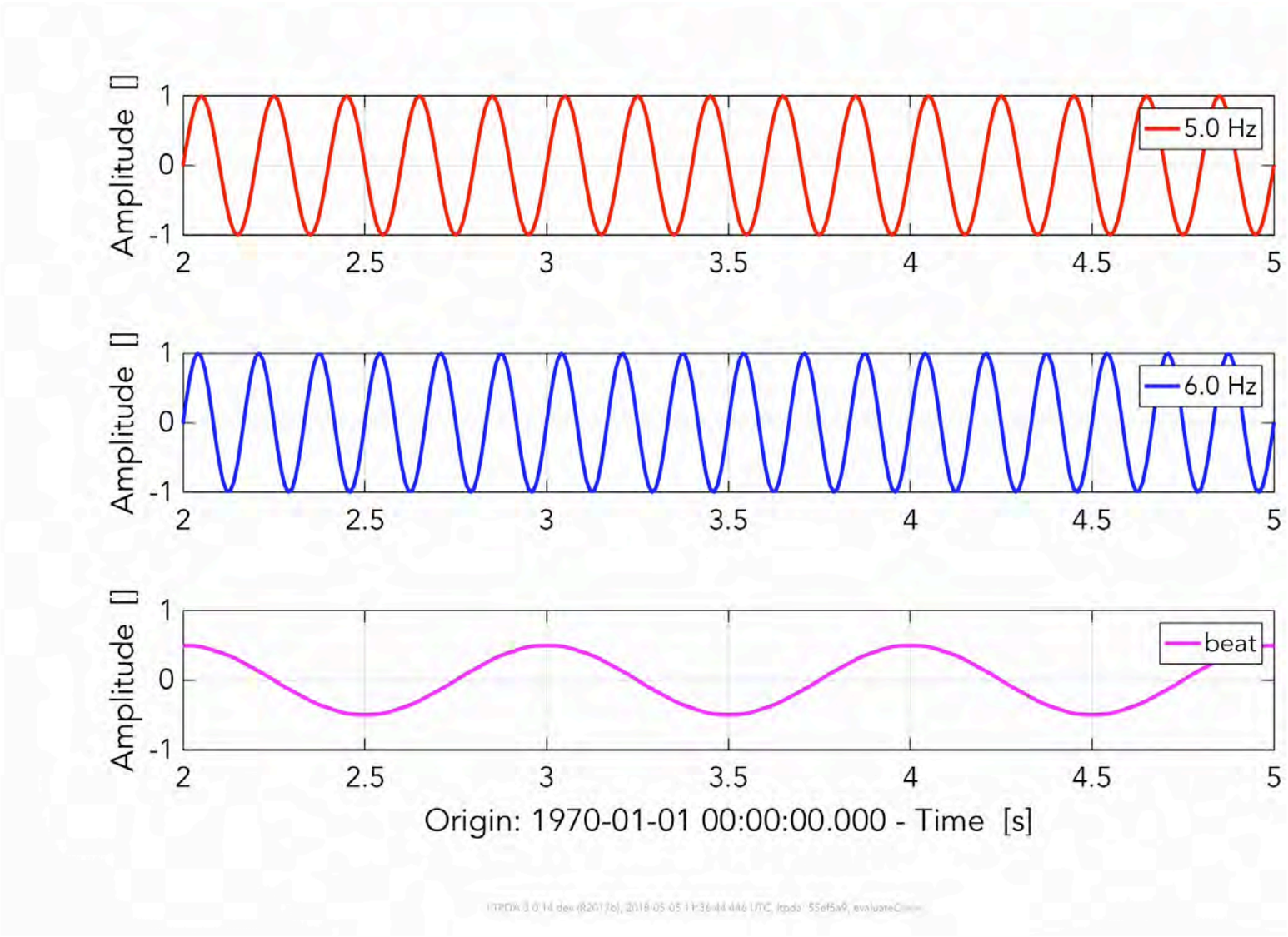
d.c. power

oscillatory term at difference frequency

phase difference between two interfering beams

Beat signal in heterodyne interferometer

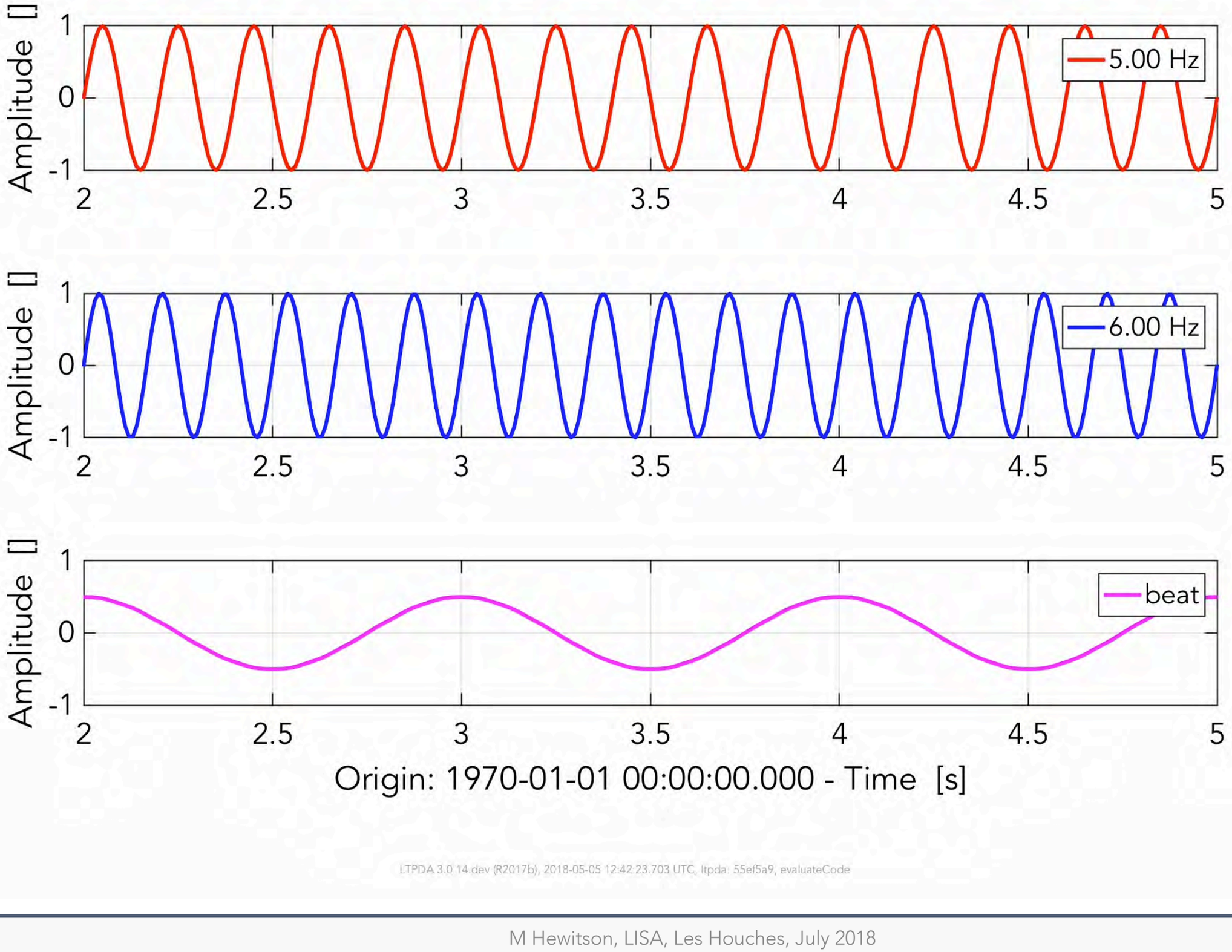
if one beam path contains a moving surface, the phase of that beam will shift and the motion will be imprinted on the phase of the beam signal



Frequency noise in heterodyne interferometers

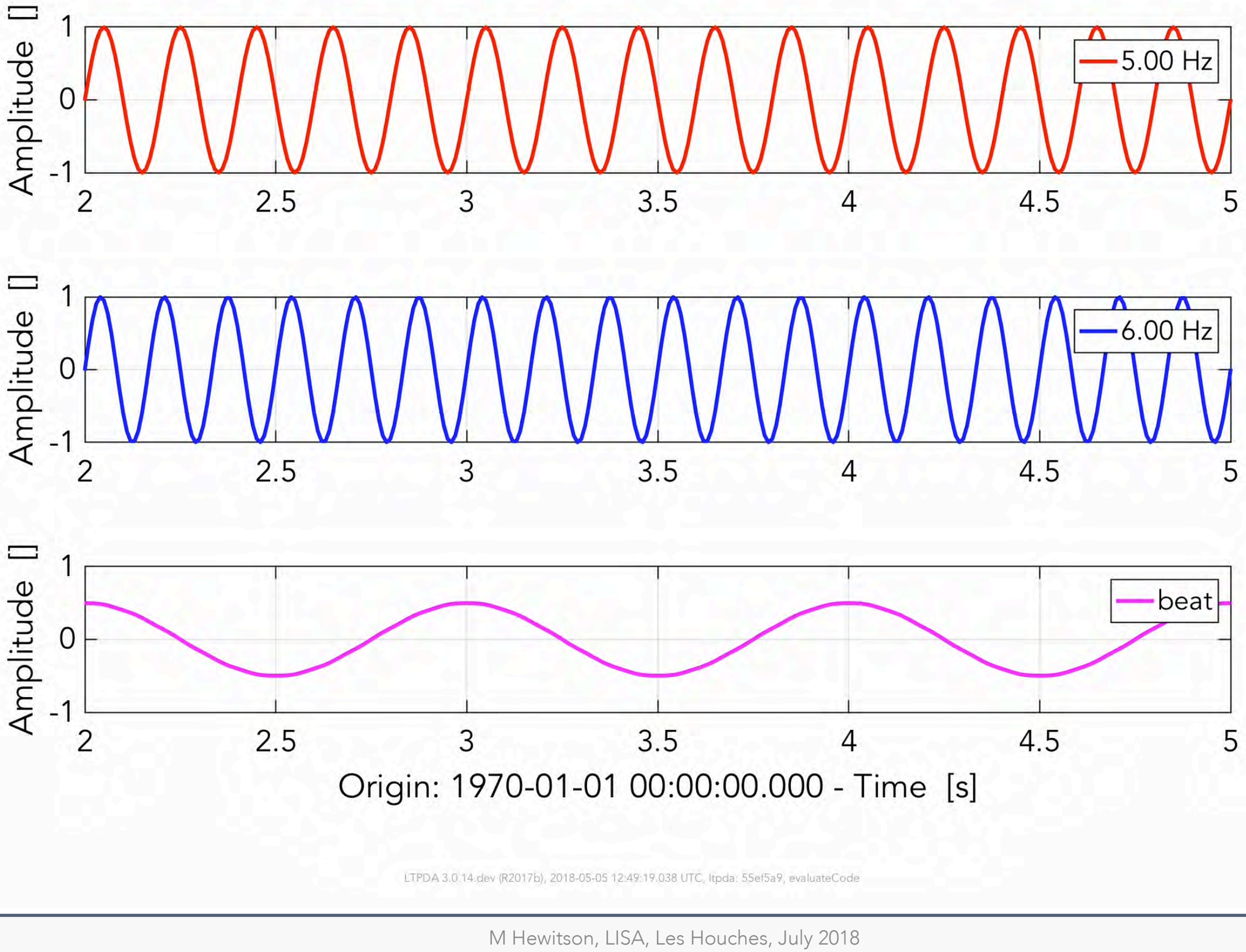
- how does frequency noise couple?
- Remember: we are measuring the phase of the beat between two laser beams
- If the frequency of one of the lasers changes, then the frequency of the measured beat will change
  - indistinguishable from a phase shift $f = \dot{\phi}$

Frequency changes in one beam



LTPDA 3.0.14.dev (R2017b), 2018-05-05 12:42:23.703 UTC, ltpda: 55ef5a9, evaluateCode

Same frequency changes in both beams



LTPDA 3.0.14.dev (R2017b), 2018-05-05 12:49:19.038 UTC, ltpda: 55ef5a9, evaluateCode

Frequency to phase

changes in frequency only couple to the output phase if they are different in each beam, or if the beams travel different paths

path-length difference between two beams

$$\phi(t) = \frac{2\pi\Delta L}{c}\nu(t)$$

phase

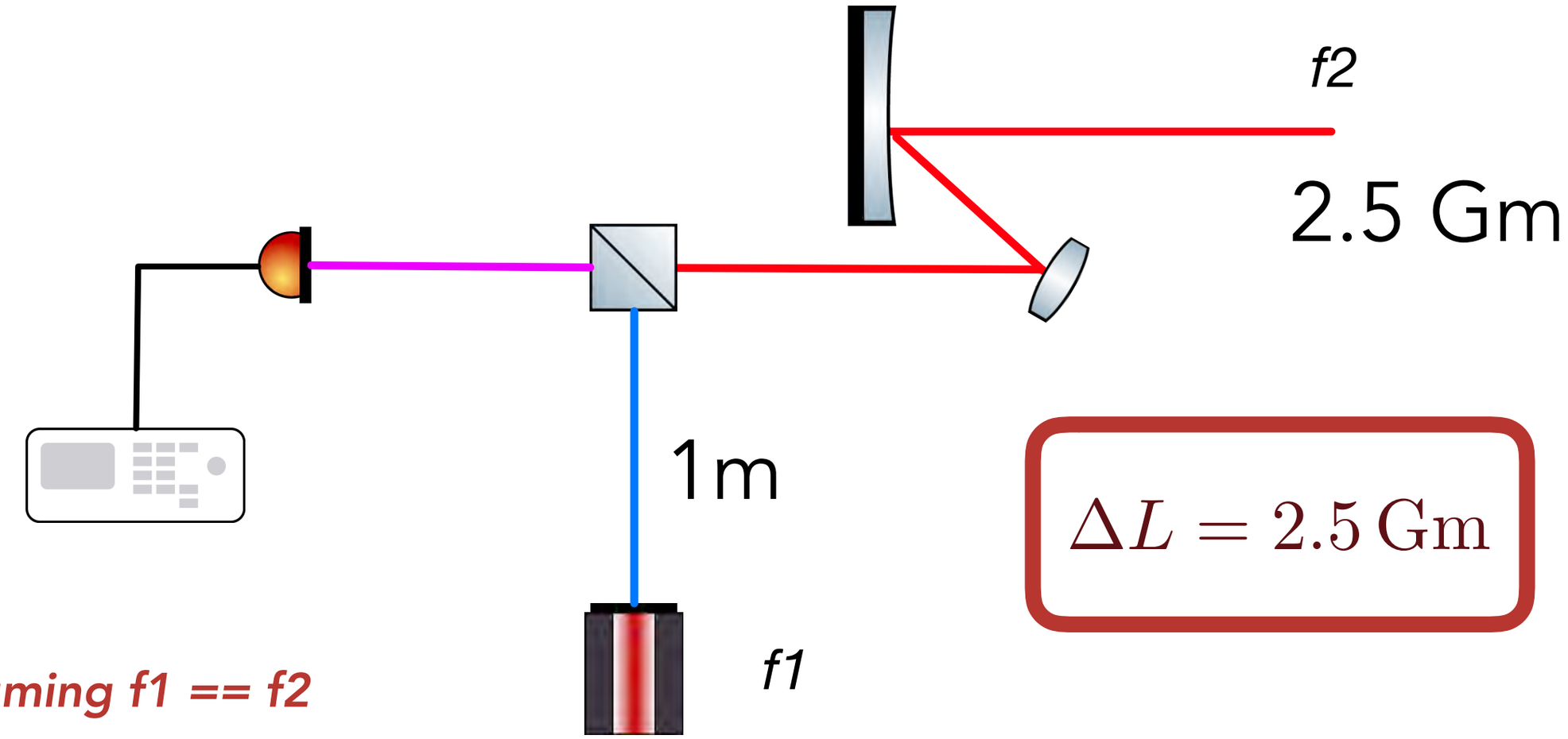
frequency

$\phi = 2\pi\nu t$

$t = \delta L/c$

Some numbers for LISA…

$$\phi(t) = \frac{2\pi\Delta L}{c}\nu(t)$$



$$\Delta L = 2.5\,\mathrm{Gm}$$

Some numbers for LISA…

$$\tilde{\phi}(f) = \frac{2\pi\Delta L}{c}\tilde{\nu}(f)$$

phase noise

frequency noise

$$\tilde{\nu}(f) = 100\ \frac{\mathrm{Hz}}{\sqrt{\mathrm{Hz}}}$$

Some numbers for LISA...

$$\tilde{\phi}(f) = \frac{2\pi\Delta L}{c}\tilde{\nu}(f)$$

$$\tilde{\phi}(f) = \frac{2\pi 2.5\times 10^{9}[\mathrm{m}]}{3\times 10^{8}[\mathrm{m}]}\times 100\frac{\mathrm{Hz}}{\sqrt{\mathrm{Hz}}}$$

$$= 5236\frac{\mathrm{rad}}{\sqrt{\mathrm{Hz}}}$$

convert phase to equivalent displacement

$$x = \frac{\lambda}{2\pi}\phi$$

$$\tilde{x} = \frac{\lambda}{2\pi}\tilde{\phi}$$

$$= \frac{1064[\mathrm{nm}]}{2\pi}5236\frac{\mathrm{rad}}{\sqrt{\mathrm{Hz}}}$$

$$= 0.9\frac{\mathrm{mm}}{\sqrt{\mathrm{Hz}}}$$

Does that work?

- We aim for a strain sensitivity of around

$$h = \frac{\Delta L}{L} = 10^{-20} \frac{1}{\sqrt{\mathrm{Hz}}}$$

- That means we need to measure distance changes around $\Delta L = 2.5 \times 10^{9} [\mathrm{m}] \times 10^{-20} \frac{1}{\sqrt{\mathrm{Hz}}}$

$$= 25 \frac{\mathrm{pm}}{\sqrt{\mathrm{Hz}}}$$

- But frequency noise is around $\tilde{x}^{\mathrm{freq}} = 0.9 \frac{\mathrm{mm}}{\sqrt{\mathrm{Hz}}}$

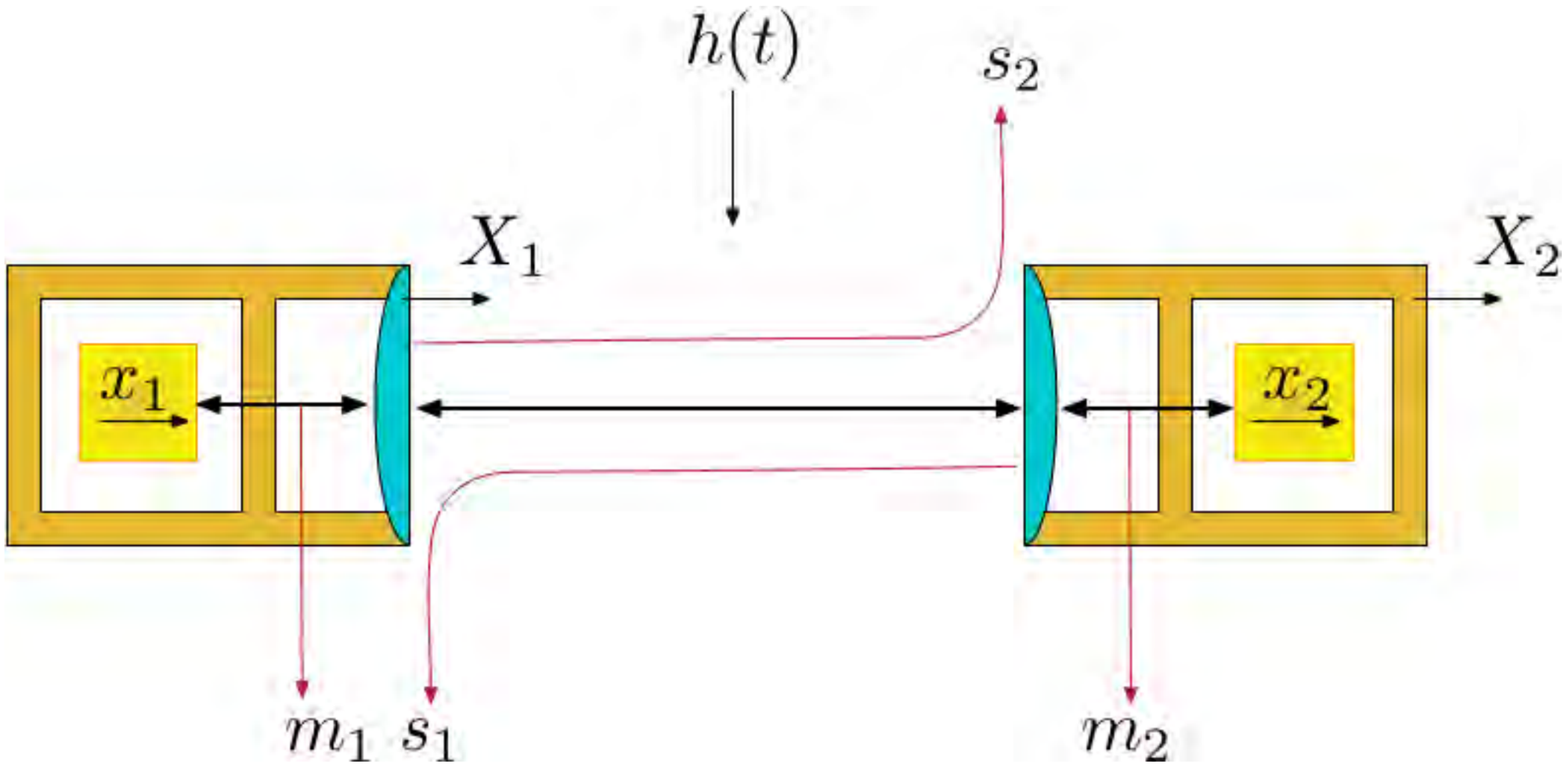
?

Cancelling frequency noise

- We need to do something to cancel the frequency noise that's introduced in the long-arm interferometer

- Use a concept called **T**ime **D**elay **I**nterferometery

Measurements in one arm

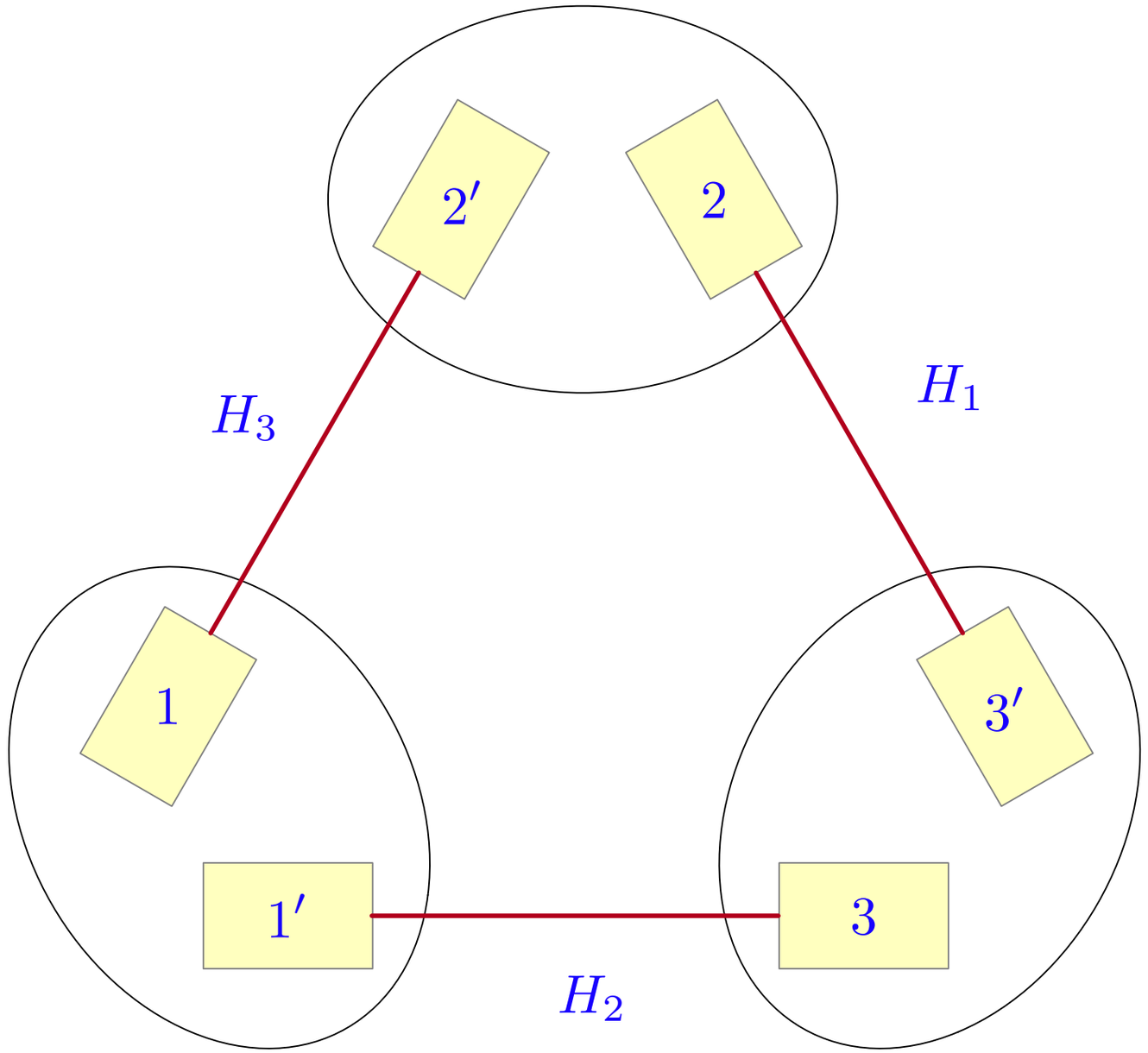


LISA measurements and TDI

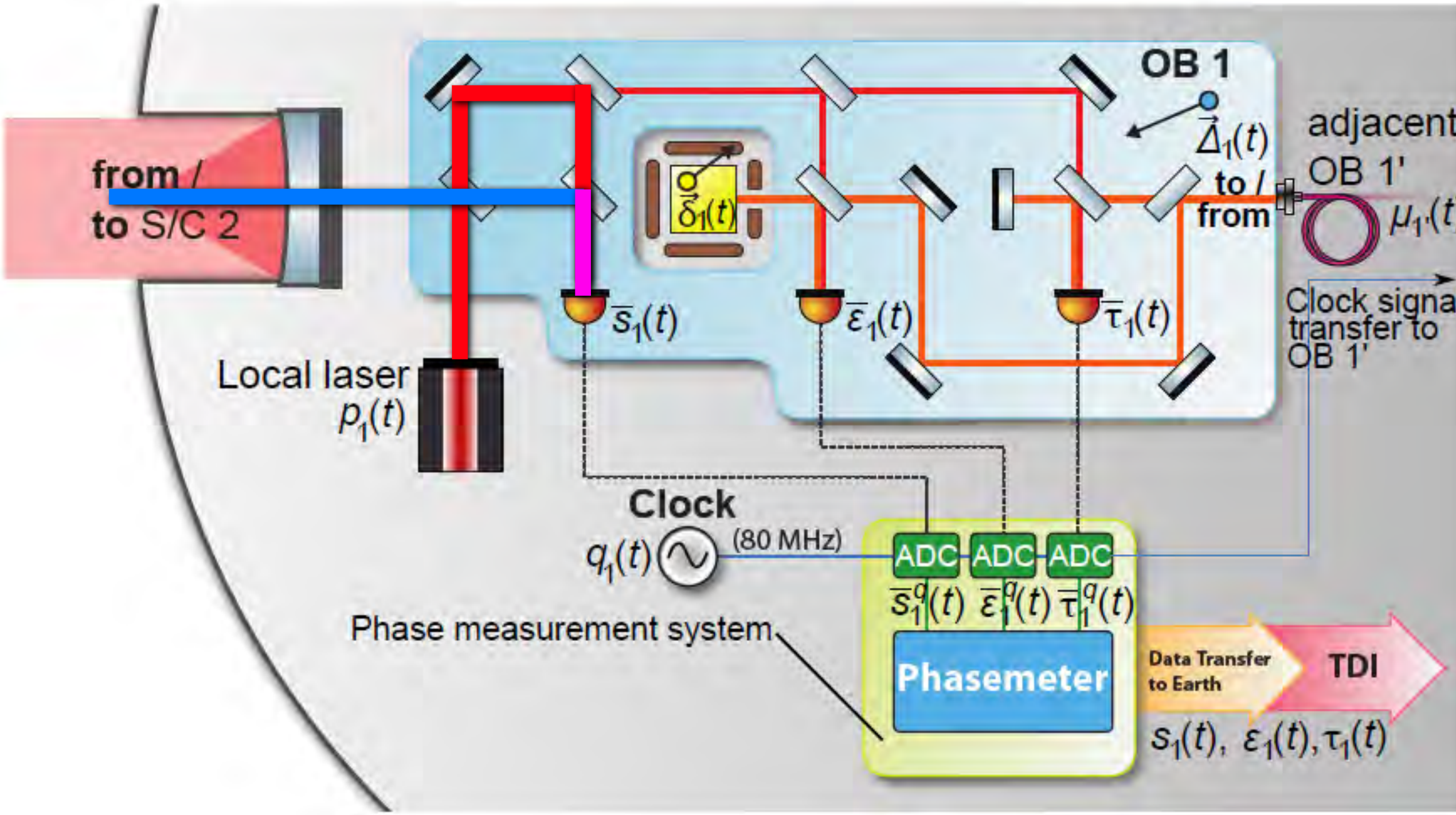
- What follows next is vastly simplified, just to get the idea across
- More detailed (correct) derivations are in:
  - Otto, Time-Delay Interferometry Simulations for LISA, PhD Thesis 2015
  - Tinto & Dhurandhar, Time-Delay Interferometry, Living Reviews, Dec 2005
    - https://link.springer.com/article/10.12942/lrr-2005-4

All measurements and notation

- Each S/C contains two optical benches
  - *X* and *X*′
- Arms are labelled as opposite vertex



Measurements in one S/C



Measurements: Science Interferometer(s)

path-length change due to GW signal

path-length change due phase noise in local S/C

path-length changes due to motion of local S/C

$$s_1 = H_3 + \mathcal{D}(p_{2'}) - p_1 - \mathcal{D}(\Delta_{2'}) - \Delta_1 + N_1$$

measured path-length change due to phase difference between local laser and received light

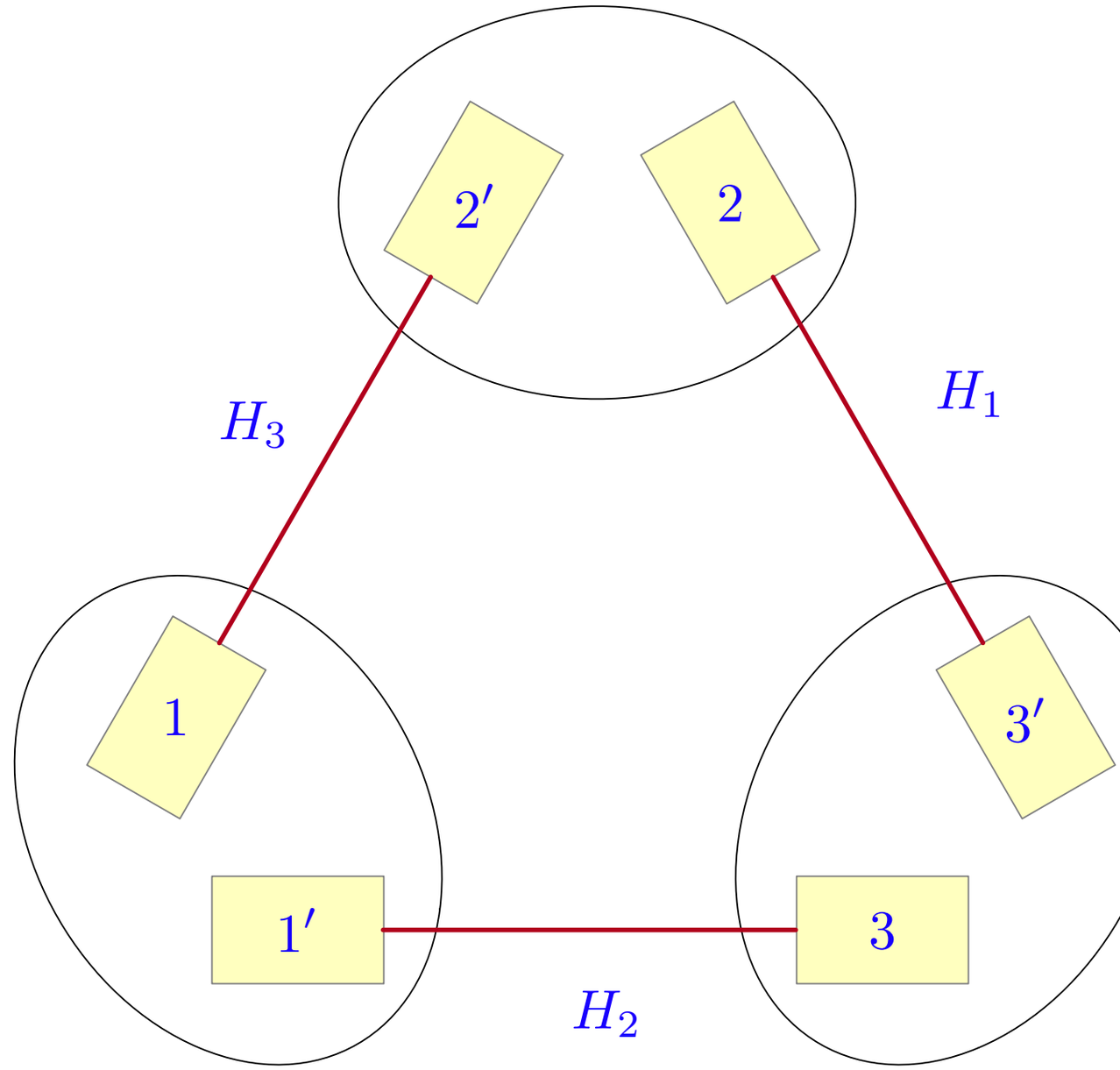
path-length change due to phase noise of far S/C (delayed)

path-length changes due to motion of far S/C (delayed)

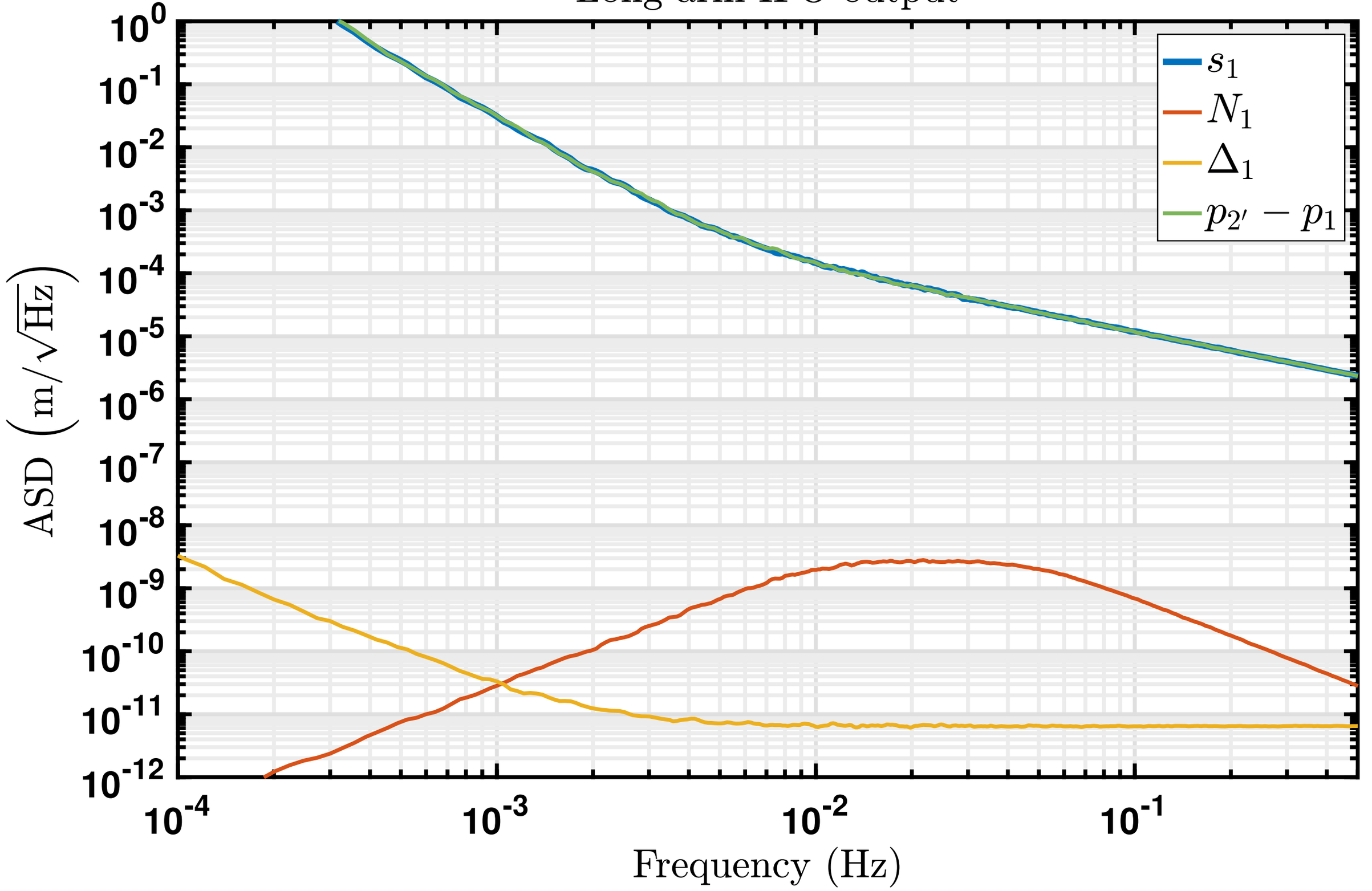
Noise in IFO readout, e.g. shot noise

What's the delay?

- For this simple treatment, arms are all assumed to be the same length and not varying in time
- That means each arm imparts the same delay on the information travelling along the arm
- We indicate this delay as an operator like this $\mathcal{D}(x)$
- And multiple delays like this $\mathcal{D}^2(x) = \mathcal{D}(\mathcal{D}(x))$

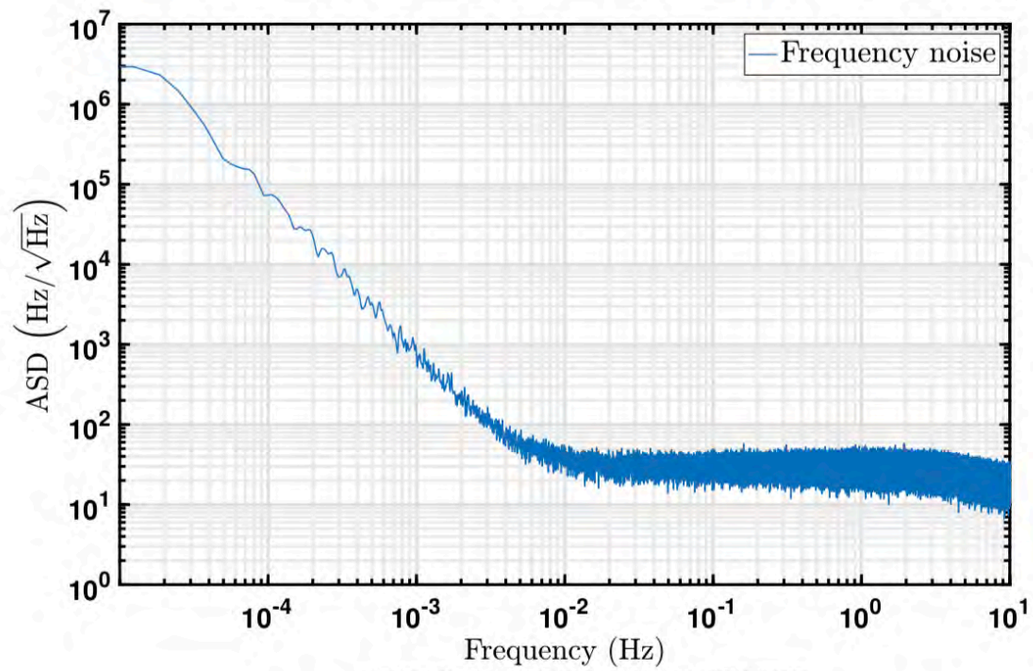


Simulation


Long-arm IFO output
s_1
N_1
Δ_1
$p_{2'} - p_1$
ASD $\left(\mathrm{m}/\sqrt{\mathrm{Hz}}\right)$
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
Frequency (Hz)

Aside: simulating the signals

- Simulate time-series of fundamental noise sources using your favourite tools
- E.g. Frequency noise:

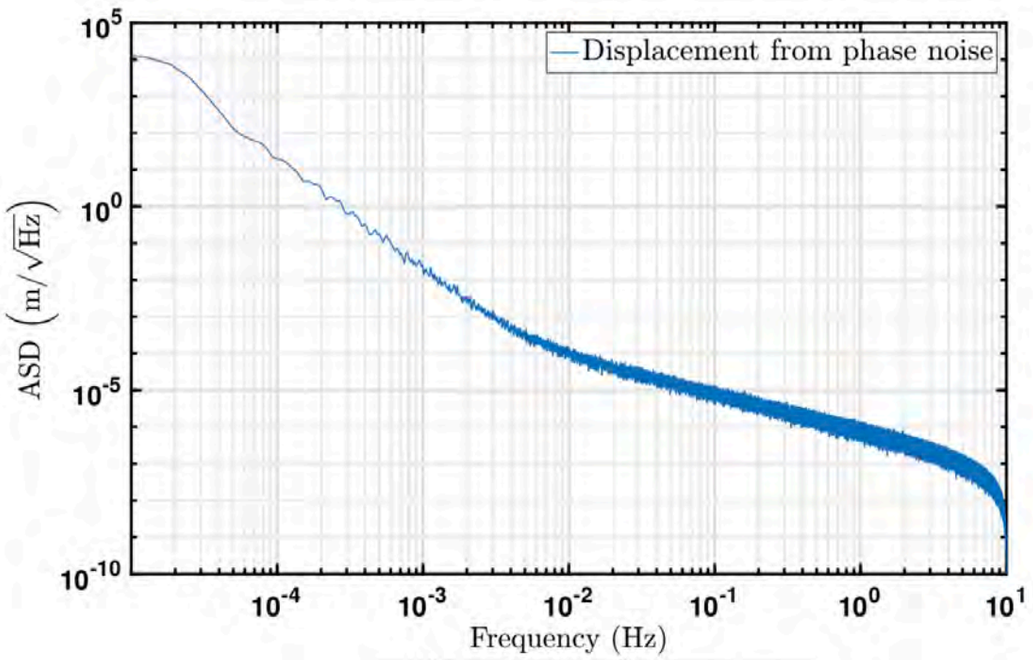


- Phase noise is then given by

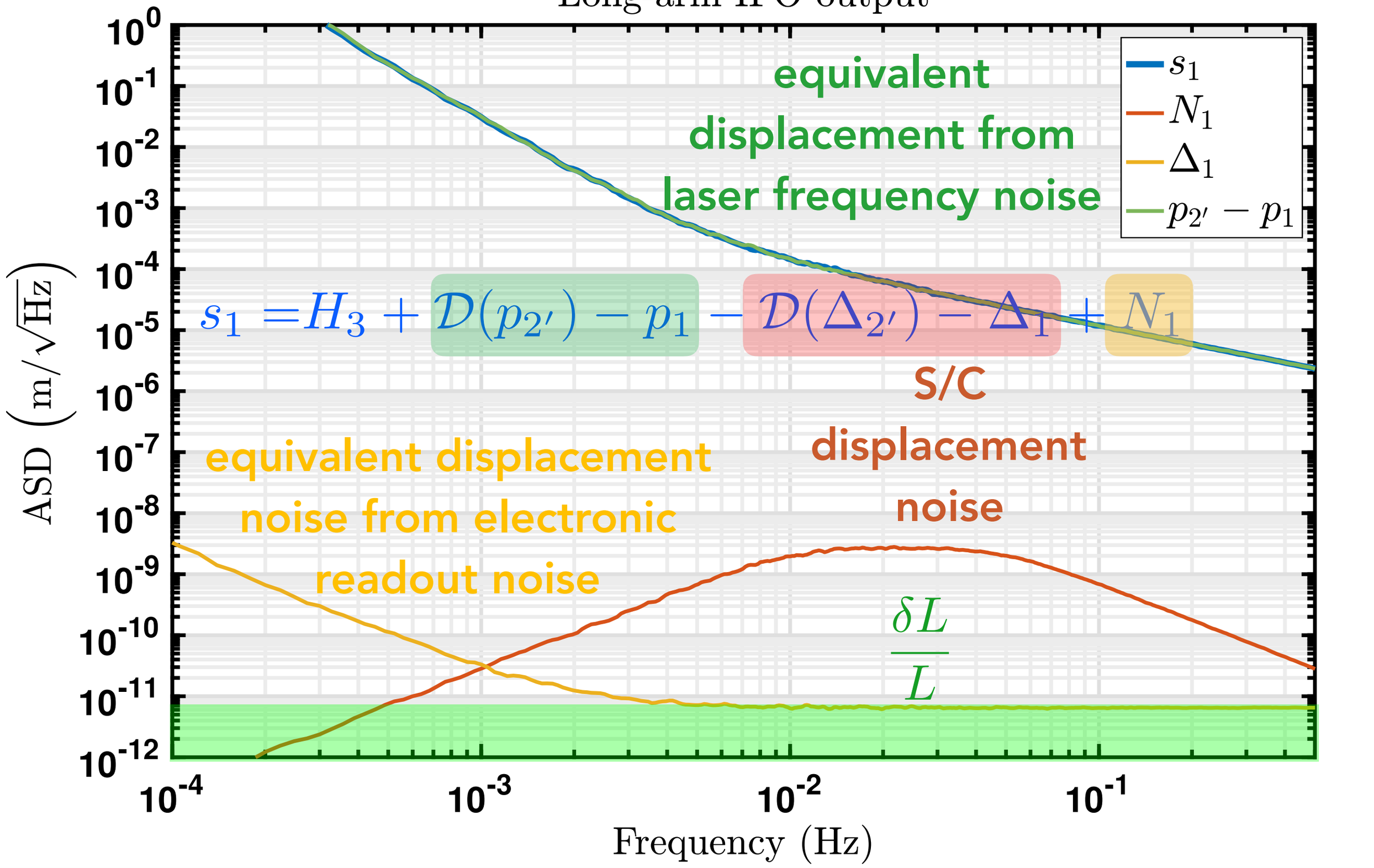
$$p(t) = \int \nu(t) \mathrm{d}t$$

- and equivalent displacement noise is then

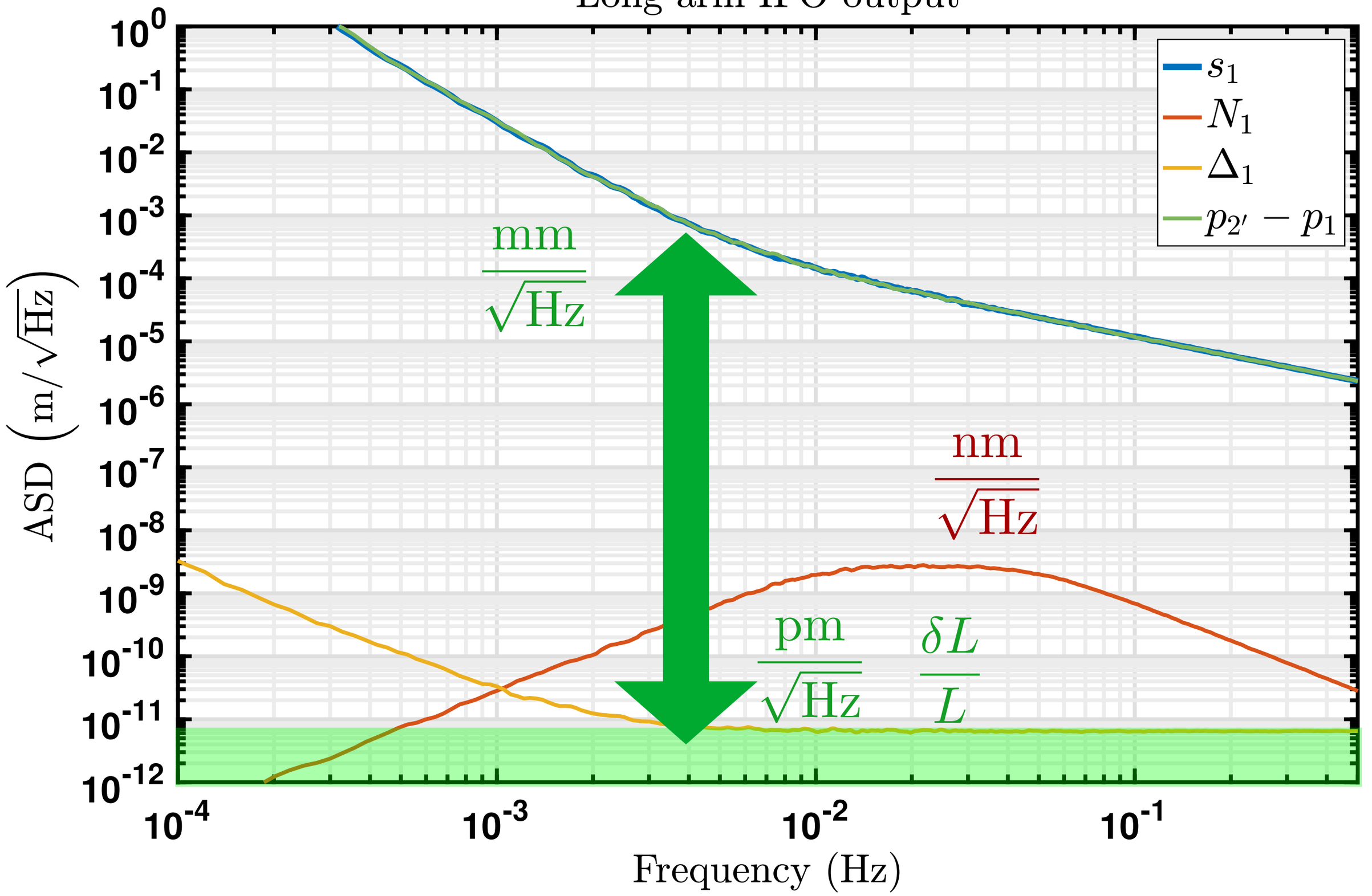
$$x_p(t) = \frac{\lambda}{2\pi} p(t)$$



Signal composition



Long-arm measurement



Measurements: Reference Interferometer

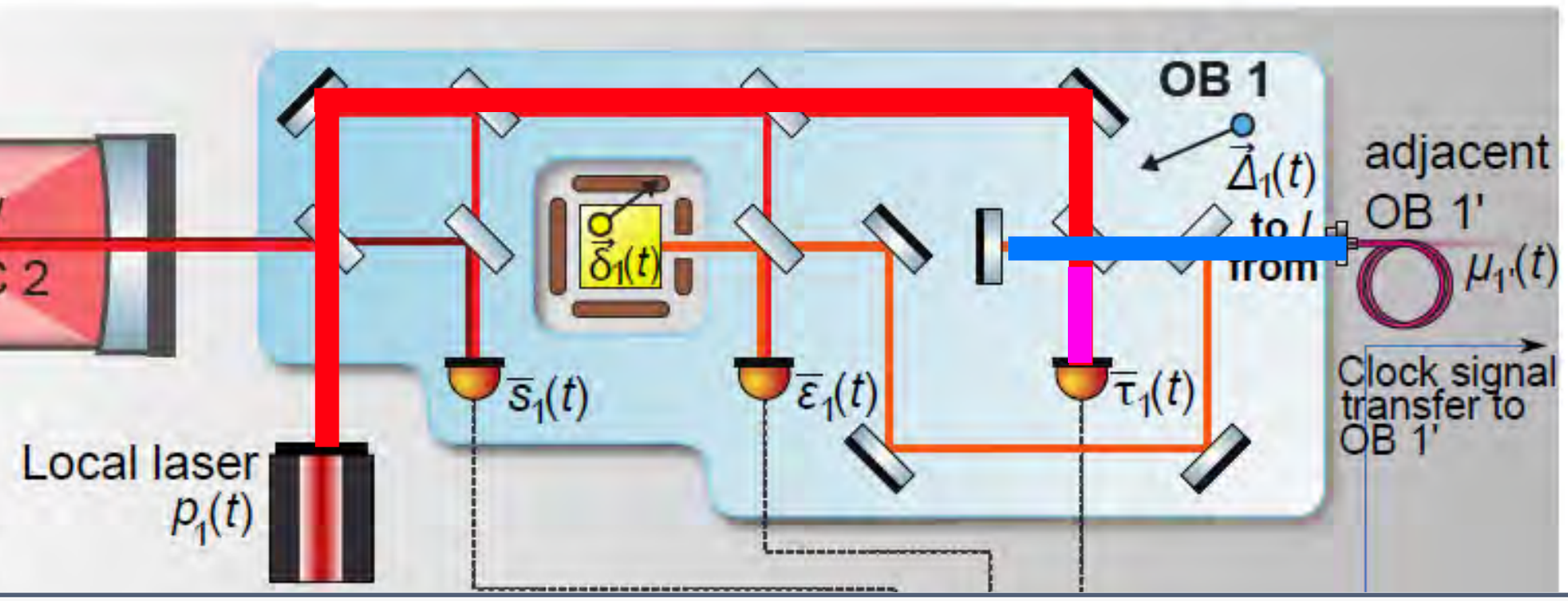
$$\tau_1 = p_{1'} - p_1 + \mu_1$$

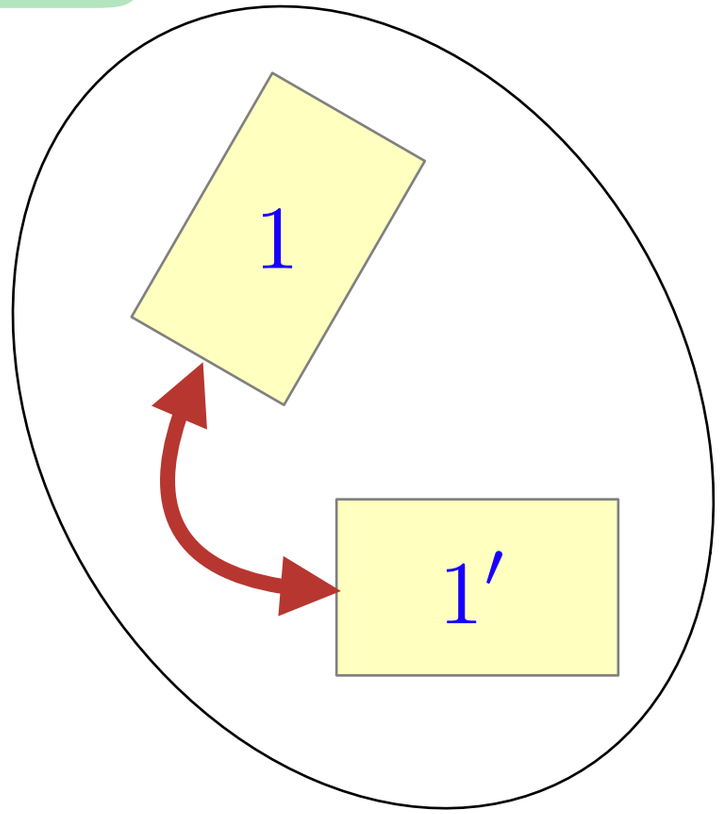
phase noise of laser 1

phase noise of back-link fibre

phase difference between two lasers in S/C 1

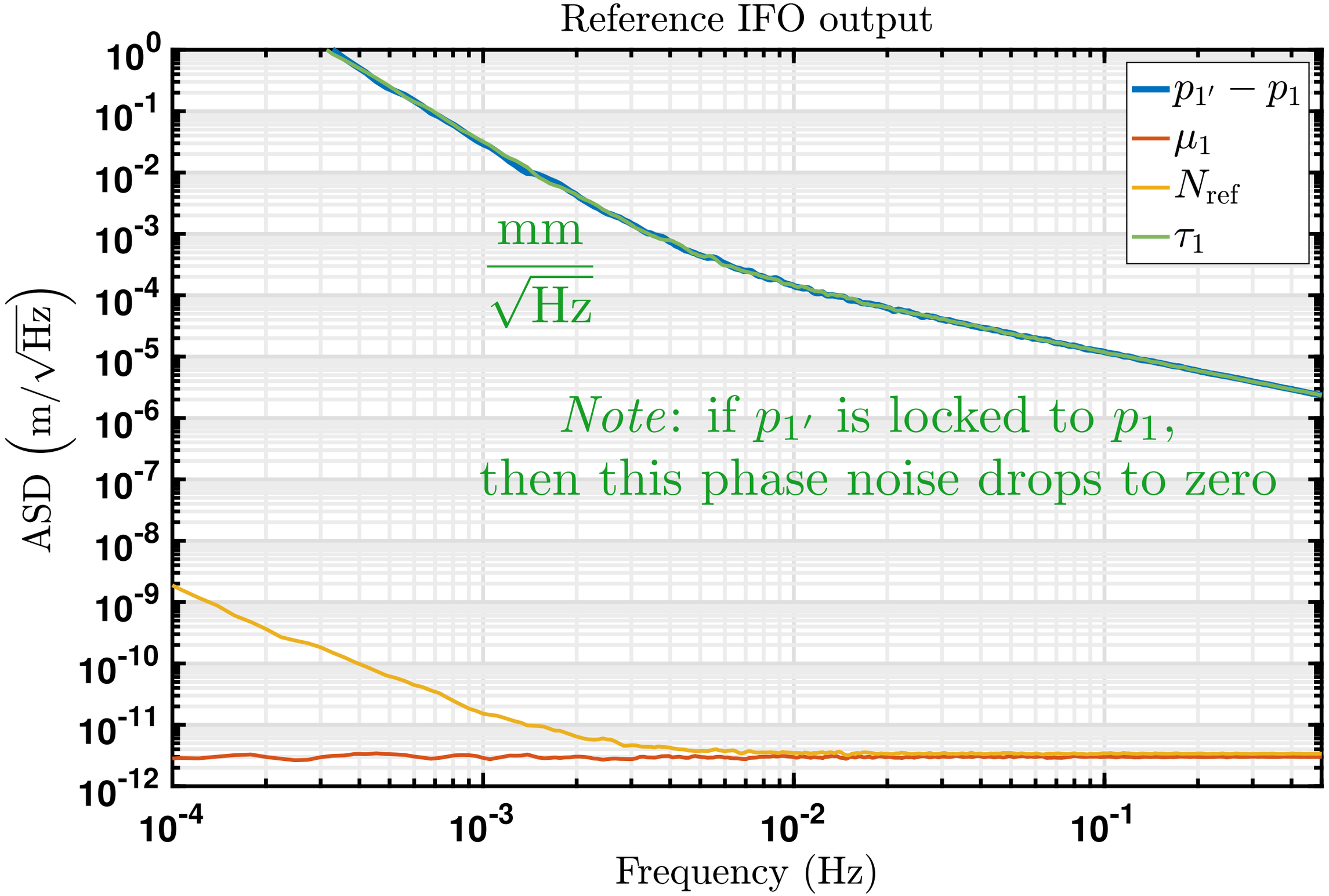
phase noise of laser 1′





back-link transports light from one bench to the other

Simulation of reference IFO



LTPDA 3.0.14.dev (R2017b), 2018-07-18 14:56:22.273 UTC, ltpda: 55ef5a9, evaluateCode

Measurements: Test mass Interferometer

phase noise of laser 1′

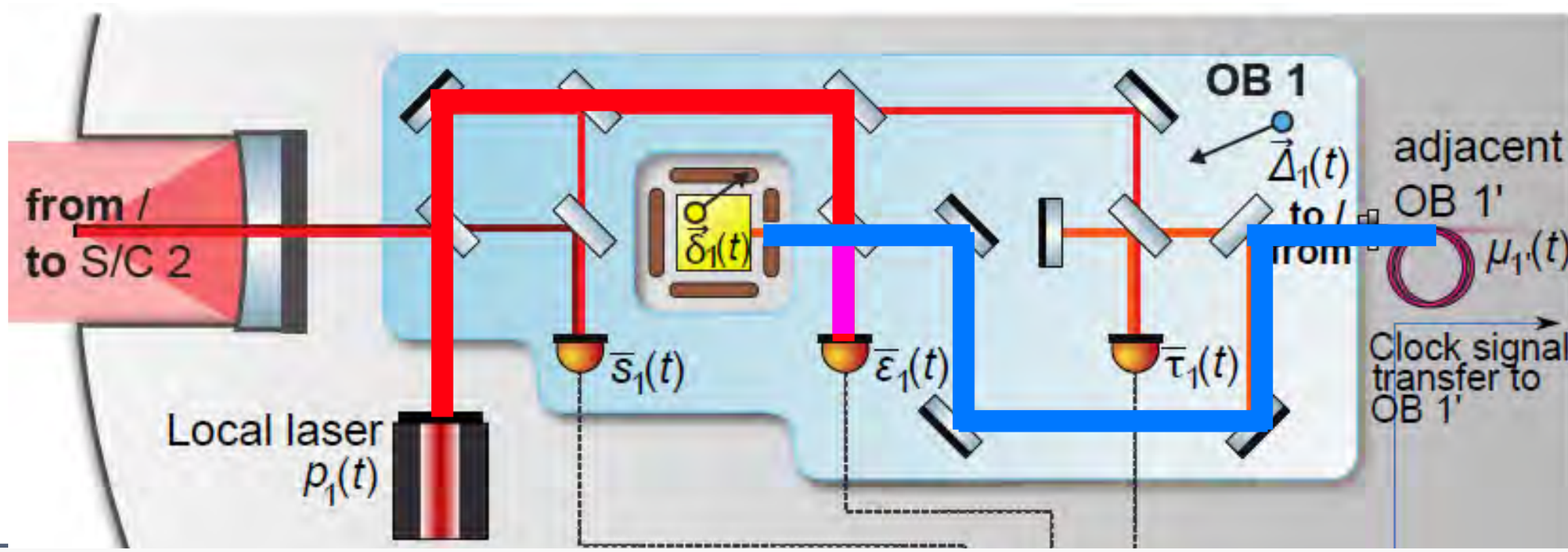
phase noise of back-link

phase change due to relative motion of TM1 and S/C

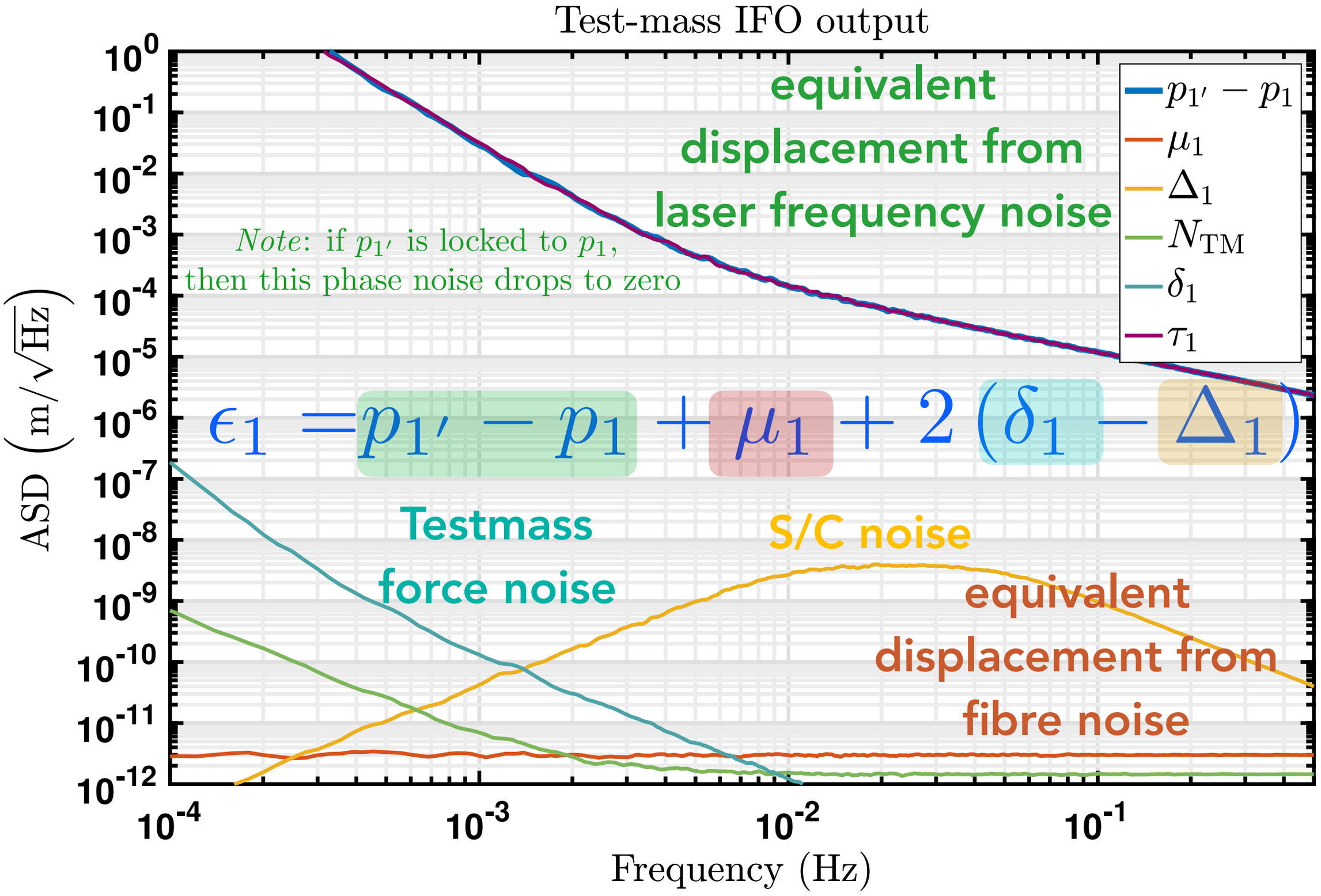
$$\epsilon_1 = p_{1'} - p_1 + \mu_1 + 2\left(\delta_1 - \Delta_1\right)$$

phase path-length change due to local laser reflected off test mass and adjacent laser delivered over the back-link

phase noise of laser 1



Simulation of test-mass IFO

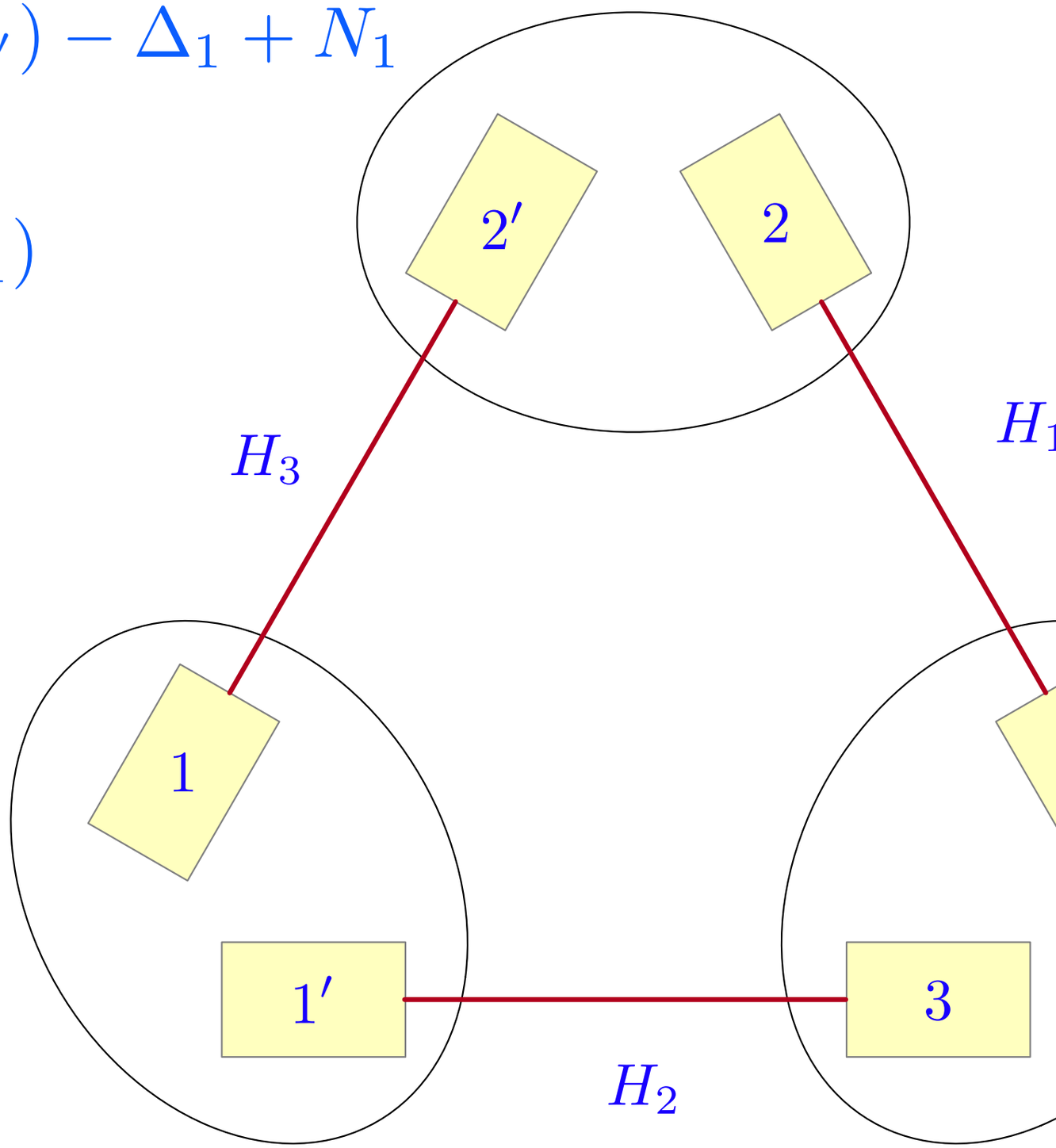


Summary for OB1

$$s_1 = H_3 + \mathcal{D}(p_{2'}) - p_1 - \mathcal{D}(\Delta_{2'}) - \Delta_1 + N_1$$

$$\tau_1 = p_{1'} - p_1 + \mu_1$$

$$\epsilon_1 = p_{1'} - p_1 + \mu_1 + 2\,(\delta_1 - \Delta_1)$$

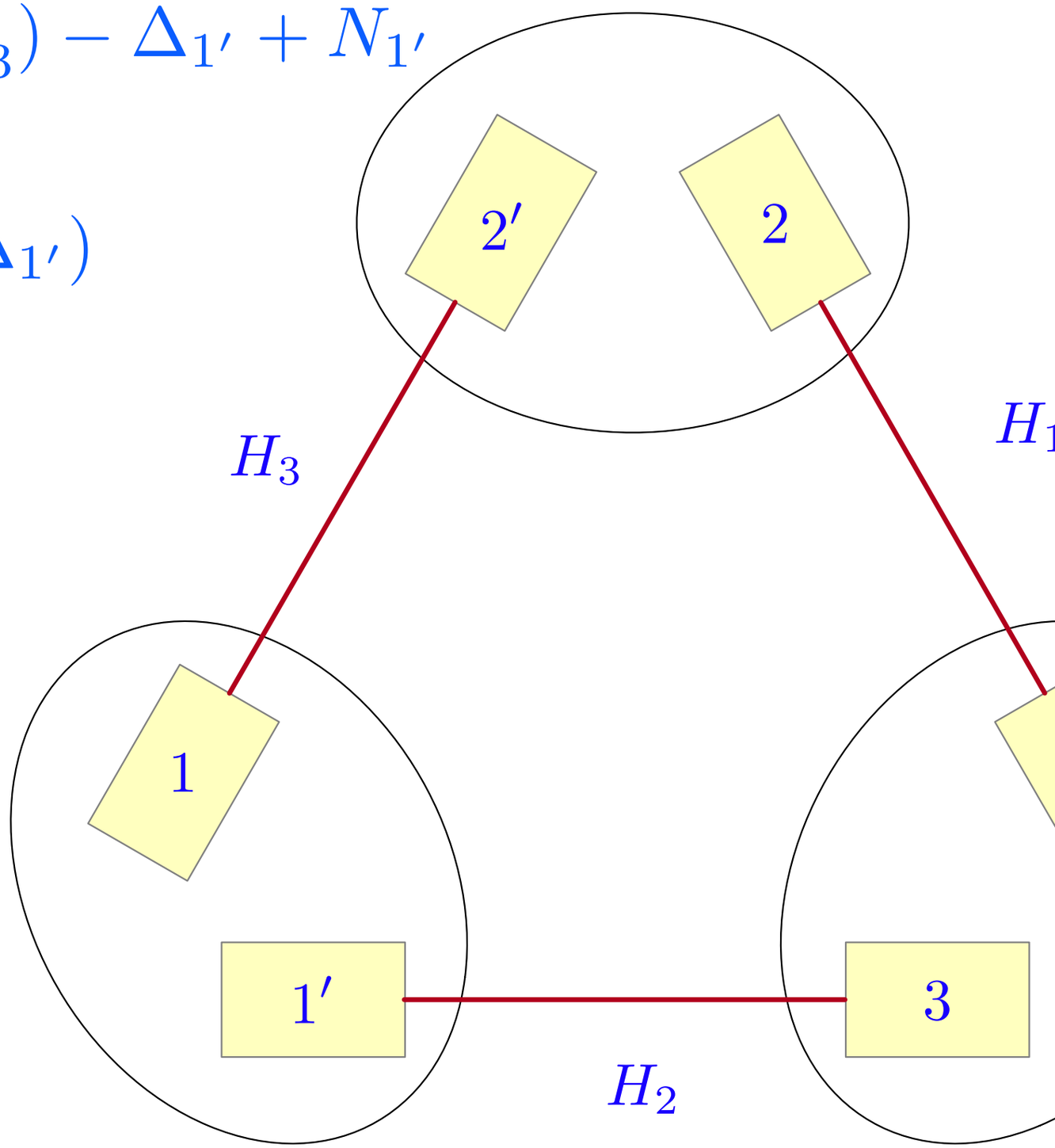


Similarly for the adjacent optical bench

$$s_{1'} = H_2 + \mathcal{D}(p_3) - p_{1'} - \mathcal{D}(\Delta_3) - \Delta_{1'} + N_{1'}$$

$$\tau_{1'} = p_1 - p_{1'} + \mu_1$$

$$\epsilon_{1'} = p_1 - p_{1'} + \mu_1 + 2\left(\delta_{1'} - \Delta_{1'}\right)$$



Cyclic permutation is your friend!

$$1 \to 2, 2 \to 3, 3 \to 1$$
$$1' \to 2', 2' \to 3', 3' \to 1'$$

$$s_2 = H_1 + \mathcal{D}(p_{3'}) - p_2 - \mathcal{D}(\Delta_{3'}) - \Delta_2 + N_2$$
$$\tau_2 = p_{2'} - p_2 + \mu_2$$
$$\epsilon_2 = p_{2'} - p_2 + \mu_2 + 2\left(\delta_2 - \Delta_2\right)$$

$$s_{2'} = H_3 + \mathcal{D}(p_1) - p_{2'} - \mathcal{D}(\Delta_1) - \Delta_{2'} + N_{2'}$$
$$\tau_{2'} = p_2 - p_{2'} + \mu_2$$
$$\epsilon_{2'} = p_2 - p_{2'} + \mu_2 + 2\left(\delta_{2'} - \Delta_{2'}\right)$$

Cyclic permutation is your friend!

$$
\begin{aligned}
s_1 &= H_3 + \mathcal{D}(p_{2'}) - p_1 - \mathcal{D}(\Delta_{2'}) - \Delta_1 + N_1 \\
\tau_1 &= p_{1'} - p_1 + \mu_1 \\
\epsilon_1 &= p_{1'} - p_1 + \mu_1 + 2\,(\delta_1 - \Delta_1) \\
s_{1'} &= H_2 + \mathcal{D}(p_3) - p_{1'} - \mathcal{D}(\Delta_3) - \Delta_{1'} + N_{1'} \\
\tau_{1'} &= p_1 - p_{1'} + \mu_1 \\
\epsilon_{1'} &= p_1 - p_{1'} + \mu_1 + 2\,(\delta_{1'} - \Delta_{1'}) \\
s_2 &= H_1 + \mathcal{D}(p_{3'}) - p_2 - \mathcal{D}(\Delta_{3'}) - \Delta_2 + N_2 \\
\tau_2 &= p_{2'} - p_2 + \mu_2 \\
\epsilon_2 &= p_{2'} - p_2 + \mu_2 + 2\,(\delta_2 - \Delta_2) \\
s_{2'} &= H_3 + \mathcal{D}(p_1) - p_{2'} - \mathcal{D}(\Delta_1) - \Delta_{2'} + N_{2'} \\
\tau_{2'} &= p_2 - p_{2'} + \mu_2 \\
\epsilon_{2'} &= p_2 - p_{2'} + \mu_2 + 2\,(\delta_{2'} - \Delta_{2'}) \\
s_3 &= H_2 + \mathcal{D}(p_{1'}) - p_3 - \mathcal{D}(\Delta_{1'}) - \Delta_3 + N_3 \\
\tau_3 &= p_{3'} - p_3 + \mu_3 \\
\epsilon_3 &= p_{3'} - p_3 + \mu_3 + 2\,(\delta_3 - \Delta_3) \\
s_{3'} &= H_1 + \mathcal{D}(p_2) - p_{3'} - \mathcal{D}(\Delta_2) - \Delta_{3'} + N_{3'} \\
\tau_{3'} &= p_3 - p_{3'} + \mu_3 \\
\epsilon_{3'} &= p_3 - p_{3'} + \mu_3 + 2\,(\delta_{3'} - \Delta_{3'})
\end{aligned}
$$

S/C1

S/C2

S/C3

$$1 \to 2, 2 \to 3, 3 \to 1$$

$$1' \to 2', 2' \to 3', 3' \to 1'$$

?

Cancelling S/C motion (I)

- Look at the arm from bench 1 to bench 2′
- Form a local measurement of TM1 compared to reference interferometer

$$x_1 = \frac{\epsilon_1 - \tau_1}{2}$$

- Same at the other end of the arm $x_{2'} = \frac{\epsilon_{2'} - \tau_{2'}}{2}$

Cancelling S/C motion (II)

- Remove motion of OB 1 and OB 2′ by forming:

$$\xi_1 = s_1 - x_1 - \mathcal{D}(x_{2'})$$

$$s_1 = H_3 + \mathcal{D}(p_{2'}) - p_1 - \mathcal{D}(\Delta_{2'}) - \Delta_1 + N_1$$

Cancelling S/C motion (II)

- Remove motion of OB 1 and OB 2′ by forming:

$$\begin{aligned}\xi_1 =& s_1 - x_1 - \mathcal{D}(x_{2'}) \\ =& s_1 - \frac{\epsilon_1 - \tau_1}{2} - \frac{\mathcal{D}(\epsilon_{2'}) - \mathcal{D}(\tau_{2'})}{2}\end{aligned}$$

$$s_1 = H_3 + \mathcal{D}(p_{2'}) - p_1 - \mathcal{D}(\Delta_{2'}) - \Delta_1 + N_1$$

Cancelling S/C motion (II)

- Remove motion of OB 1 and OB 2′ by forming:

$$s_1 = H_3 + \mathcal{D}(p_{2'}) - p_1 - \mathcal{D}(\Delta_{2'}) - \Delta_1 + N_1$$

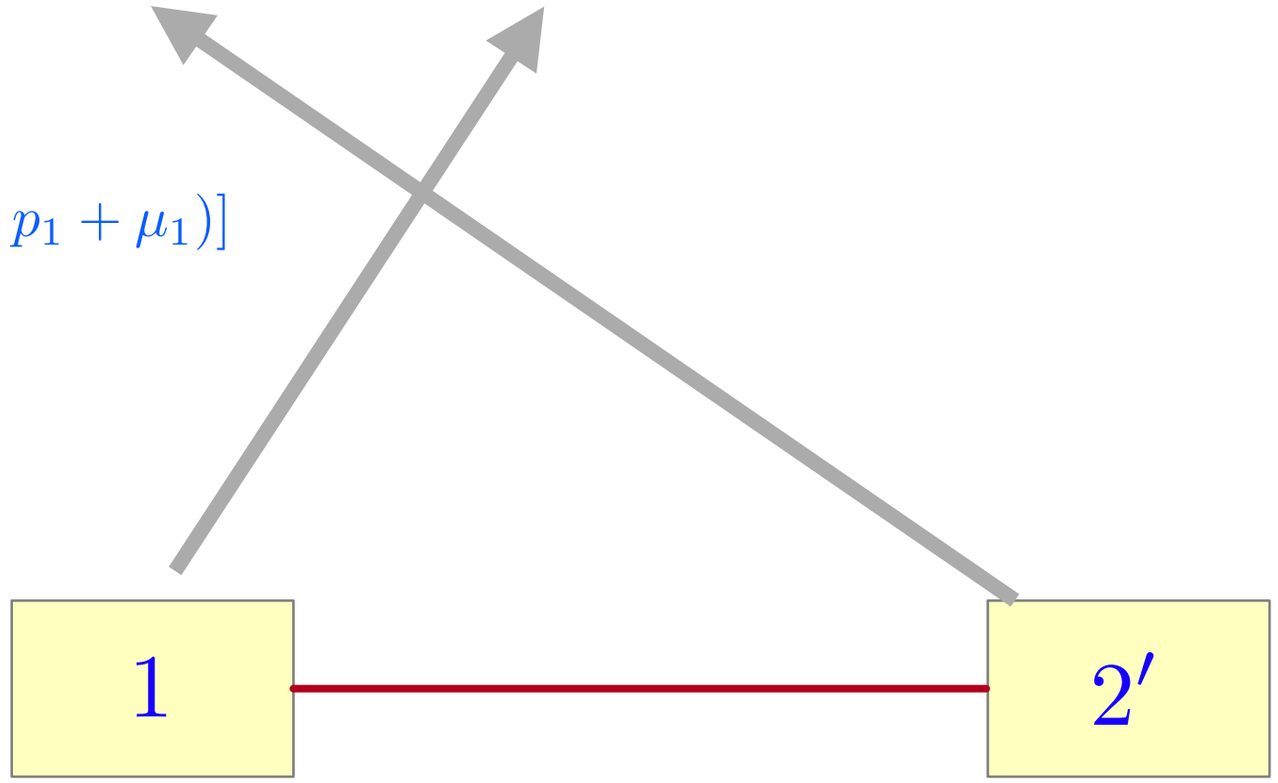
$$\xi_1 = s_1 - x_1 - \mathcal{D}(x_{2'})$$

$$= s_1 - \frac{\epsilon_1 - \tau_1}{2} - \frac{\mathcal{D}(\epsilon_{2'}) - \mathcal{D}(\tau_{2'})}{2}$$

$$= H_3 + \mathcal{D}(p_{2'}) - \mathcal{D}(\delta_{2'}) - p_1 - \delta_1 + N_1$$

$$\frac{\epsilon_{1'} - \tau_{1'}}{2} = \frac{1}{2}\left[p_{1'} - p_1 + \mu_1 + 2(\delta_1 - \Delta_1) - (p_{1'} - p_1 + \mu_1)\right]$$

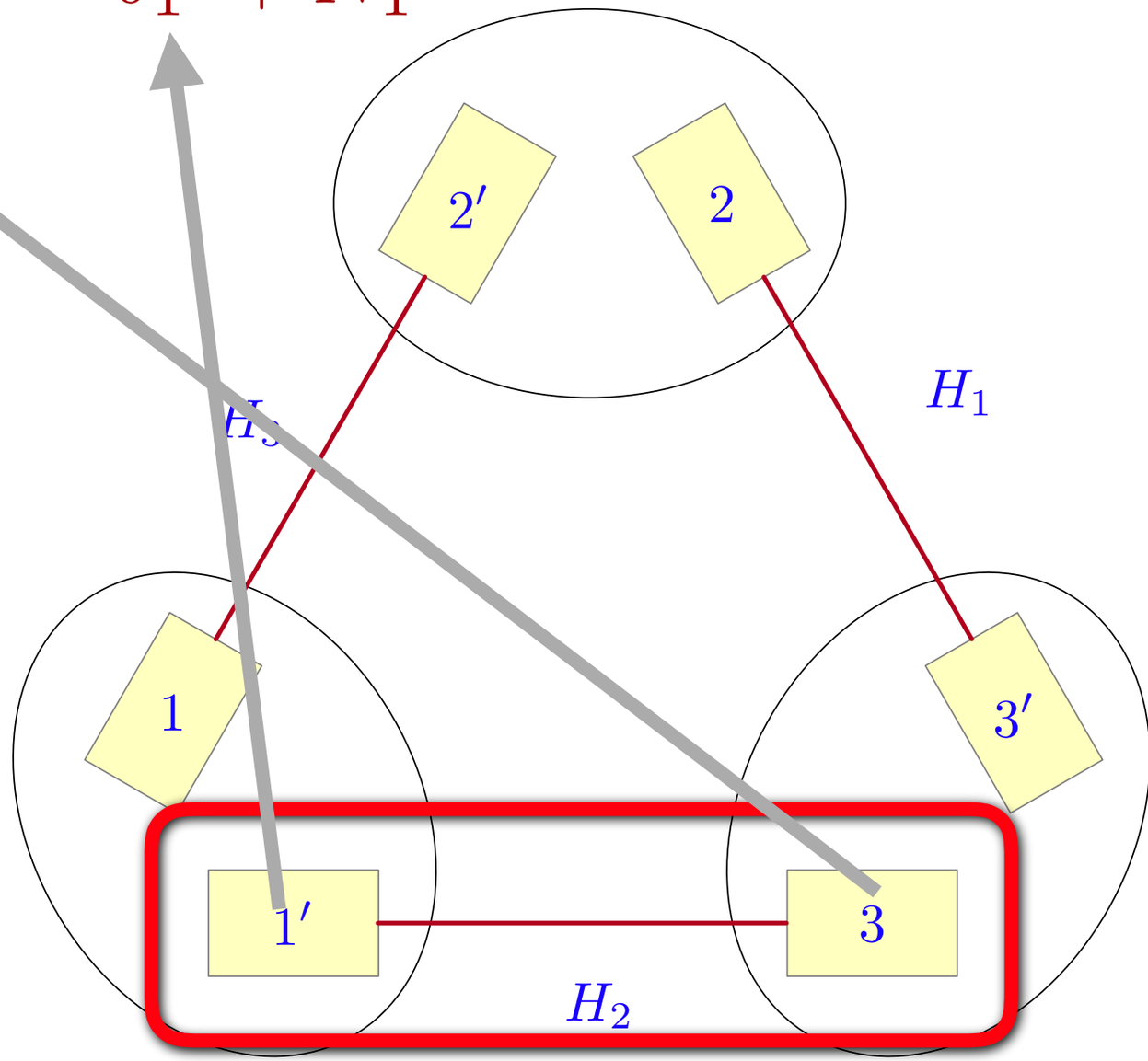
$$= \delta_1 - \Delta_1$$



Similar for adjacent arm

$$\xi_{1'} = s_{1'} - \frac{\epsilon_{1'} - \tau_{1'}}{2} - \frac{\mathcal{D}(\epsilon_3) - \mathcal{D}(\tau_3)}{2}$$

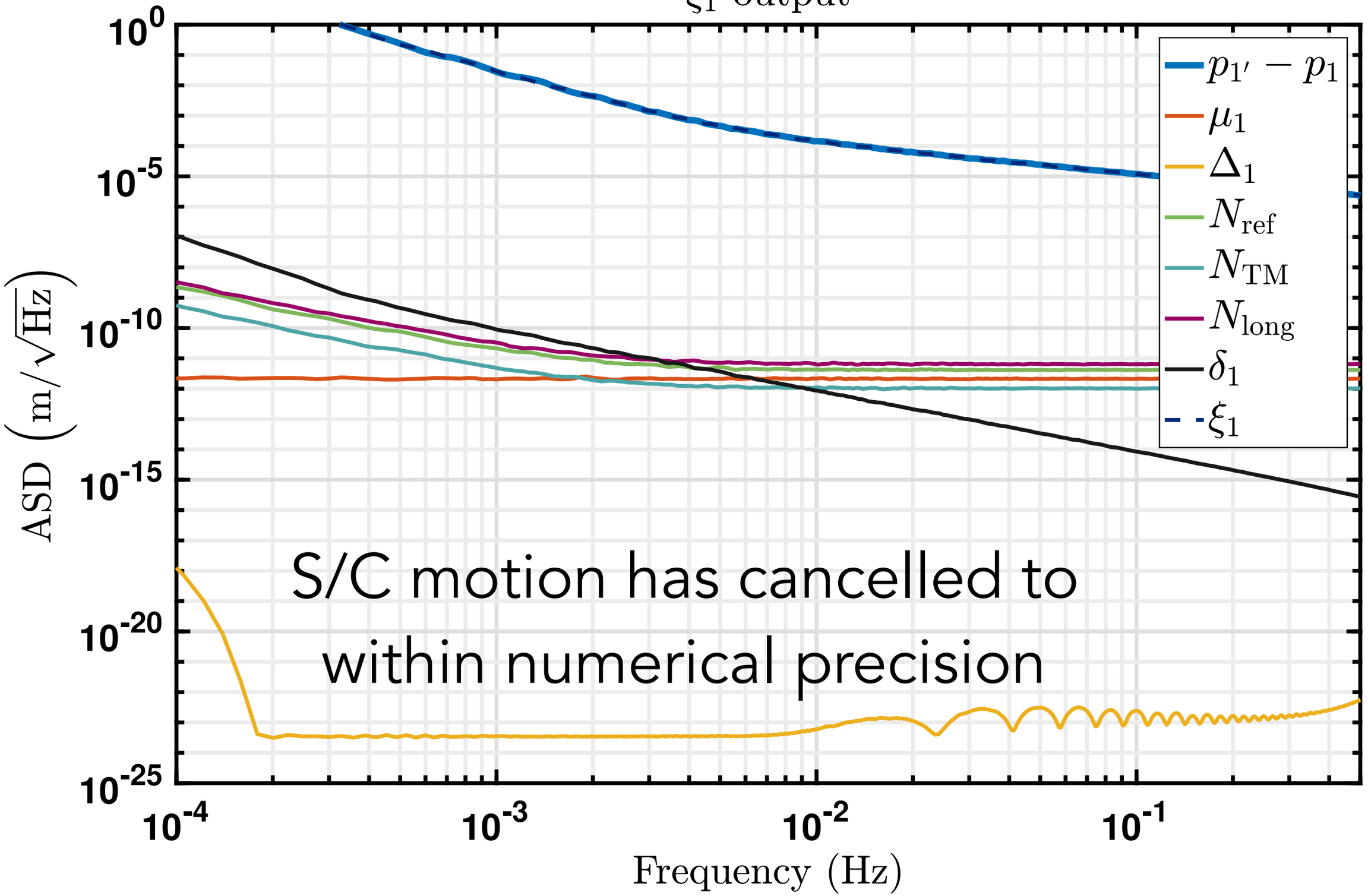
$$= s_{1'} + \mathcal{D}(p_3) - \mathcal{D}(\delta_3) - p_{1'} - \delta_{1'} + N_{1'}$$



Cyclic permutation strikes again

$$\xi_1 = H_3 + \mathcal{D}(p_{2'}) - \mathcal{D}(\delta_{2'}) - p_1 - \delta_1 + N_1$$
$$\xi_{1'} = H_2 + \mathcal{D}(p_3) - \mathcal{D}(\delta_3) - p_{1'} - \delta_{1'} + N_{1'}$$
$$\xi_2 = H_1 + \mathcal{D}(p_{3'}) - \mathcal{D}(\delta_{3'}) - p_2 - \delta_2 + N_2$$
$$\xi_{2'} = H_3 + \mathcal{D}(p_1) - \mathcal{D}(\delta_1) - p_{2'} - \delta_{2'} + N_{2'}$$
$$\xi_3 = H_2 + \mathcal{D}(p_{1'}) - \mathcal{D}(\delta_{1'}) - p_3 - \delta_3 + N_3$$
$$\xi_{3'} = H_1 + \mathcal{D}(p_2) - \mathcal{D}(\delta_2) - p_{3'} - \delta_{3'} + N_{3'}$$

Simulation


ξ_1 output
10^0
10^{-5}
10^{-10}
10^{-15}
10^{-20}
10^{-25}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
ASD $\left(\mathrm{m}/\sqrt{\mathrm{Hz}}\right)$
Frequency (Hz)
$p_{1'} - p_1$
μ_1
Δ_1
N_{ref}
N_{TM}
N_{long}
δ_1
ξ_1
S/C motion has cancelled to within numerical precision
LTPDA 3.0.14.dev (R2017b), 2018-07-18 15:49:20.770 UTC, ltpda: 55ef5a9, evaluateCode

Reduce to 3 lasers

- We can make combinations to remove the laser noise from the adjacent (primed) benches

$$\eta_1 = \xi_1 - \frac{\mathcal{D}(\tau_2)}{2} + \frac{\mathcal{D}(\tau_{2'})}{2}$$
$$= H_3 + \mathcal{D}(p_2) - p_1 - \delta_1 - \mathcal{D}(\delta_{2'}) + N_1$$

- Similar for adjacent bench

$$\eta_{1'} = H_2 + \mathcal{D}(p_3) - p_1 - \delta_{1'} - \mathcal{D}(\delta_3) + N_{1'}$$

Yes, you've guessed it...

$$\eta_1 = H_3 + \mathcal{D}(p_2) - p_1 - \delta_1 - \mathcal{D}(\delta_{2'}) + N_1$$
$$\eta_{1'} = H_2 + \mathcal{D}(p_3) - p_1 - \delta_{1'} - \mathcal{D}(\delta_3) + N_{1'}$$
$$\eta_2 = H_1 + \mathcal{D}(p_3) - p_2 - \delta_2 - \mathcal{D}(\delta_{3'}) + N_2$$
$$\eta_{2'} = H_3 + \mathcal{D}(p_1) - p_2 - \delta_{2'} - \mathcal{D}(\delta_1) + N_{2'}$$
$$\eta_3 = H_2 + \mathcal{D}(p_1) - p_3 - \delta_3 - \mathcal{D}(\delta_{1'}) + N_3$$
$$\eta_{3'} = H_1 + \mathcal{D}(p_2) - p_3 - \delta_{3'} - \mathcal{D}(\delta_2) + N_{3'}$$

TDI generators

- A TDI generator is a function of delayed linear combinations of $(\eta_1, \eta_{1'}, \eta_2, \eta_{2'}, \eta_3, \eta_{3'})$
- It results in new variables which are free of laser phase noise (frequency noise)
- Different ways of finding these solutions exist

R. W. Hellings. Elimination of clock jitter noise in spaceborne laser interferometers. Physical Review D: Particles and Fields, 64:022002, 2001.

M. Tinto, F. B. Estabrook, and J. W. Armstrong. Time-delay interferometry for LISA. Physical Review D: Particles and Fields, 65:082003, 2002.

D. A. Shaddock, M. Tinto, F. B. Estabrook, and J. W. Armstrong. Data combinations accounting for LISA spacecraft motion. Physical Review D: Particles and Fields, 68:061303(R), 2003.

TDI 1.0

- Appropriate for our case: fixed, equal arms
- There are three correlated combinations which are frequency noise free

$$X = \mathcal{D}^3(\eta_{2'}) + \mathcal{D}^2(\eta_1) + \mathcal{D}(\eta_3) + \eta_{1'} - \mathcal{D}^3(\eta_3) - \mathcal{D}^2(\eta_{1'}) - \mathcal{D}(\eta_{2'}) - \eta_1$$

$$Y = \mathcal{D}^3(\eta_{3'}) + \mathcal{D}^2(\eta_2) + \mathcal{D}(\eta_1) + \eta_1 - \mathcal{D}^3(\eta_{2'}) - \mathcal{D}^2(\eta_{3'}) - \mathcal{D}(\eta_{2'}) - \eta_1$$

$$Z = \mathcal{D}^3(\eta_{1'}) + \mathcal{D}^2(\eta_3) + \mathcal{D}(\eta_2) + \eta_{3'} - \mathcal{D}^3(\eta_2) - \mathcal{D}^2(\eta_{3'}) - \mathcal{D}(\eta_{1'}) - \eta_3$$

TDI 1.0: X

$$X = \mathcal{D}^3(\eta_{2'}) + \mathcal{D}^2(\eta_1) + \mathcal{D}(\eta_3) + \eta_{1'} - \mathcal{D}^3(\eta_3) - \mathcal{D}^2(\eta_{1'}) - \mathcal{D}(\eta_{2'}) - \eta_1$$

$$\begin{aligned}
X = \quad & \mathcal{D}^3(H_3) + \mathcal{D}^4(p_1) - \mathcal{D}^3(p_2) - \mathcal{D}^3(\delta_{2'}) - \mathcal{D}^4(\delta_1) + \mathcal{D}^3(N_{2'}) \\
& + \mathcal{D}^2(H_3) + \mathcal{D}^3(p_2) - \mathcal{D}^2(p_1) - \mathcal{D}^2\delta_1 - \mathcal{D}^3(\delta_{2'}) + \mathcal{D}^2(N_1) \\
& + \mathcal{D}(H_2) + \mathcal{D}^2(p_1) - \mathcal{D}(p_3) - \mathcal{D}(\delta_3) - \mathcal{D}^2(\delta_{1'}) + \mathcal{D}(N_3) \\
& + H_2 + \mathcal{D}(p_3) - p_1 - \delta_{1'} - \mathcal{D}(\delta_3) + N_{1'} \\
& - \mathcal{D}^3(H_2) - \mathcal{D}^4(p_1) + \mathcal{D}^3(p_3) + \mathcal{D}^3(\delta_3) + \mathcal{D}^4(\delta_{1'}) - \mathcal{D}^3(N_3) \\
& - \mathcal{D}^2(H_2) - \mathcal{D}^3(p_3) + \mathcal{D}^2(p_1) + \mathcal{D}^2(\delta_{1'}) + \mathcal{D}^3(\delta_3) - \mathcal{D}^2(N_{1'}) \\
& - \mathcal{D}(H_3) - \mathcal{D}^2(p_1) + \mathcal{D}(p_2) + \mathcal{D}(\delta_{2'}) + \mathcal{D}^2(\delta_1) - \mathcal{D}(N_{2'}) \\
& - H_3 - \mathcal{D}(p_2) + p_1 + \delta_1 + \mathcal{D}(\delta_{2'}) - N_1
\end{aligned}$$

ouch,.... just keep going...

TDI 1.0: X

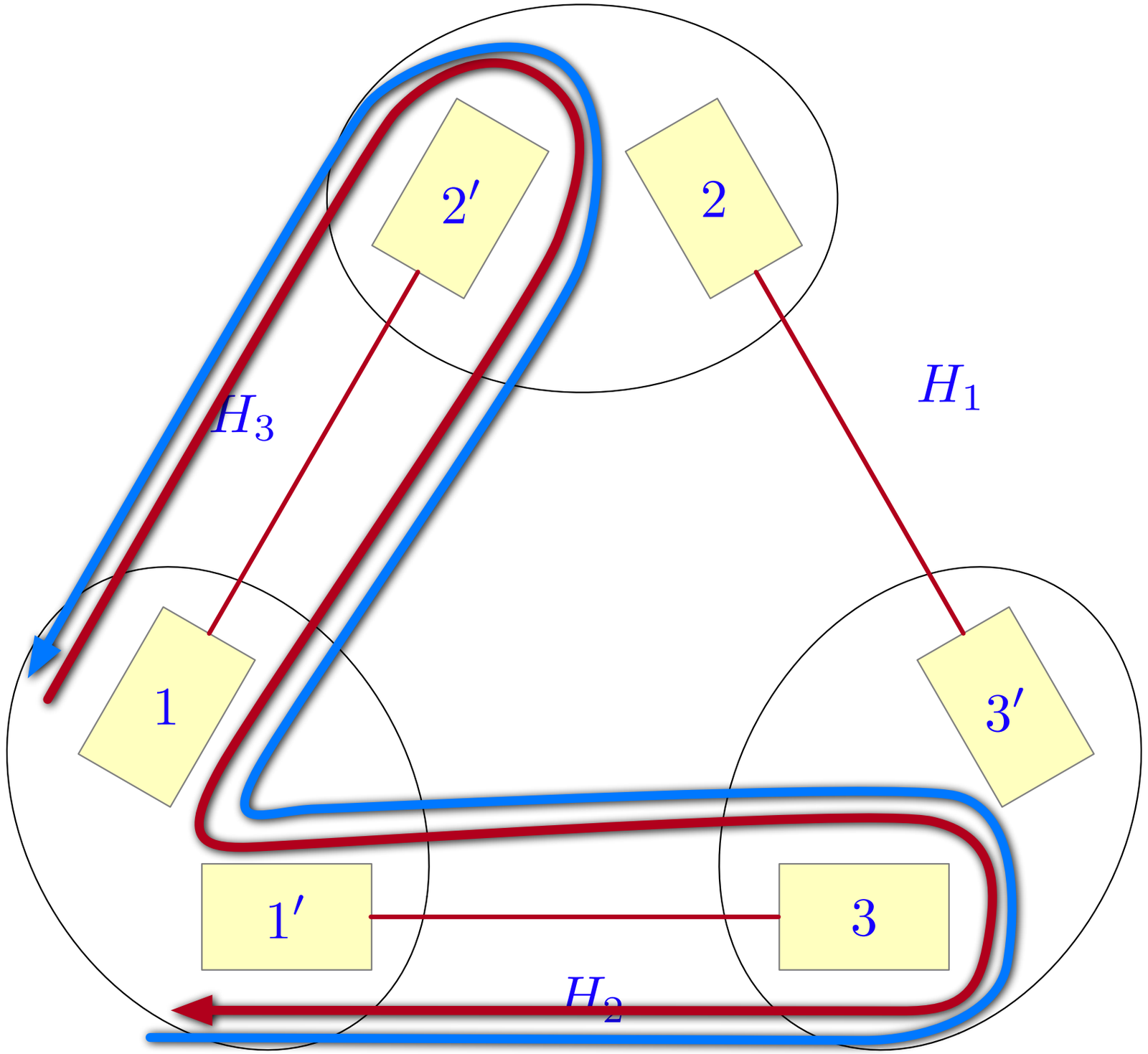
some time later…

$$
\begin{aligned}
X = \quad & \mathcal{D}^3(H_3) - \mathcal{D}^3(\delta_{2'}) - \mathcal{D}^4(\delta_1) + \mathcal{D}^3(N_{2'}) \\
& + \mathcal{D}^2(H_3) - \mathcal{D}^2(\delta_1) - \mathcal{D}^3(\delta_{2'}) + \mathcal{D}^2(N_1) \\
& + \mathcal{D}(H_2) - \mathcal{D}(\delta_3) - \mathcal{D}^2(\delta_{1'}) + \mathcal{D}(N_3) \\
& + H_2 - \delta_{1'} - \mathcal{D}(\delta_3) + N_{1'} \\
& - \mathcal{D}^3(H_2) + \mathcal{D}^3(\delta_3) + \mathcal{D}^4(\delta_{1'}) - \mathcal{D}^3(N_3) \\
& - \mathcal{D}^2(H_2) + \mathcal{D}^2(\delta_{1'}) + \mathcal{D}^3(\delta_3) - \mathcal{D}^2(N_{1'}) \\
& - \mathcal{D}(H_3) + \mathcal{D}(\delta_{2'}) + \mathcal{D}^2(\delta_1) - \mathcal{D}(N_{2'}) \\
& - H_3 + \delta_1 + \mathcal{D}(\delta_{2'}) - N_1
\end{aligned}
$$

8 x GW signal 16 x TM motion 8 x Readout noise

Test mass motion

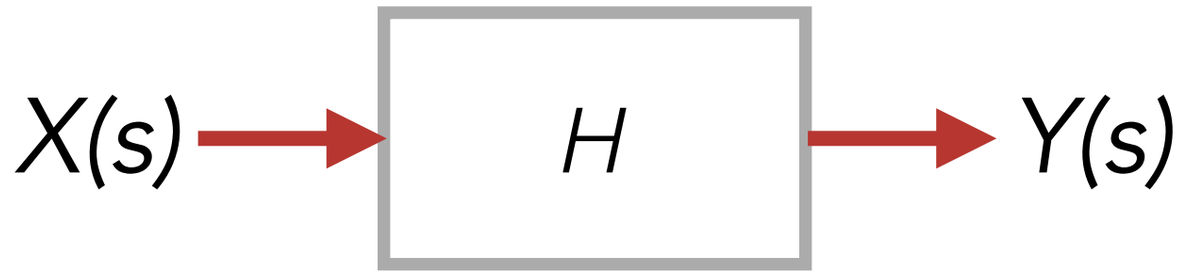
- 4 x TM 1′
- 4 x TM 1
- 4 x TM 2′
- 4 x TM 3



Transfer function

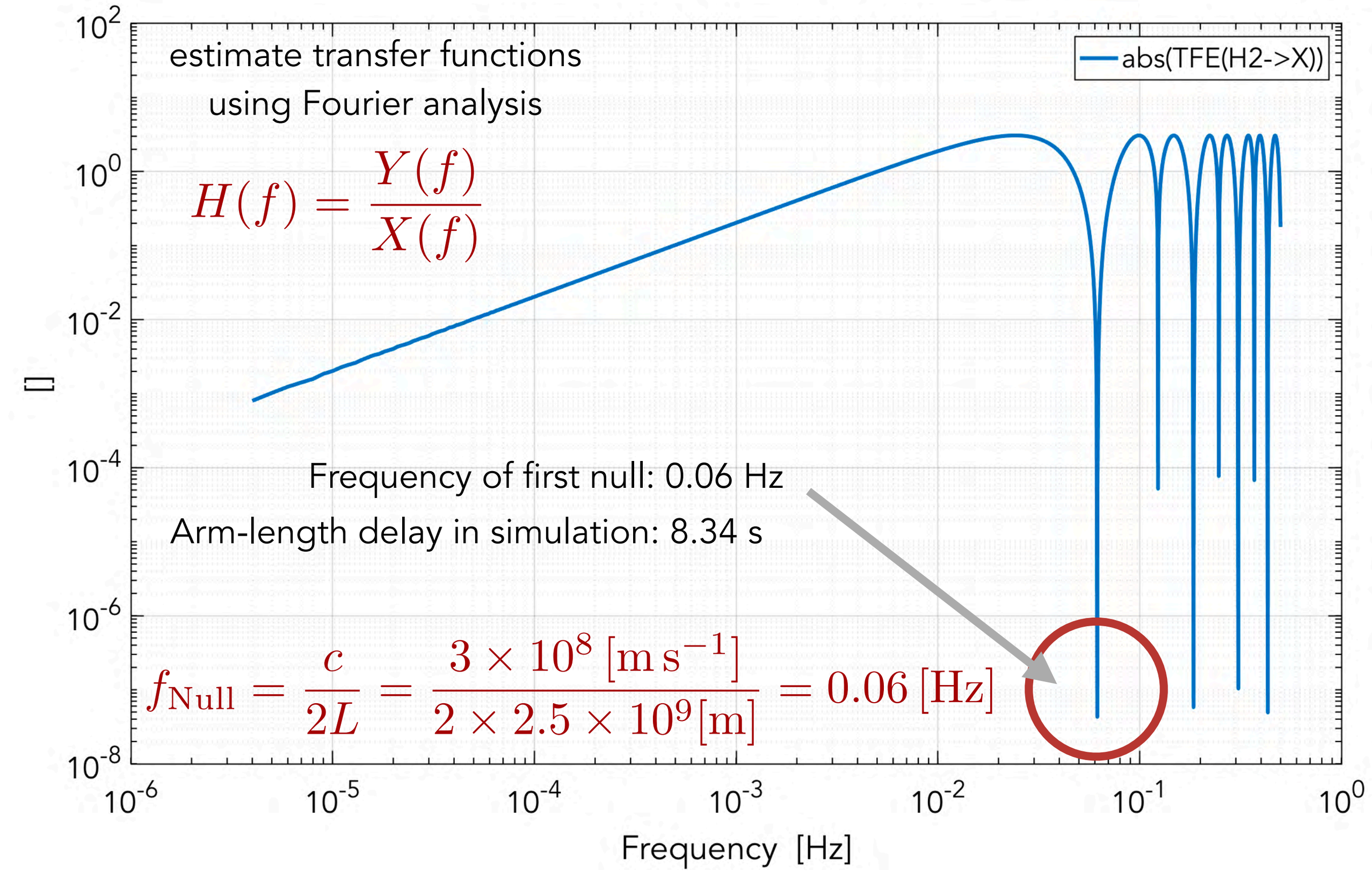
- Transfer function relates output values of a function/system to the input values

$$H(s) = \frac{Y(s)}{X(s)}$$



- Defined in Laplace Domain
  - complex function of complex variables

Transfer function: *H* to *X*



Plug in some noise

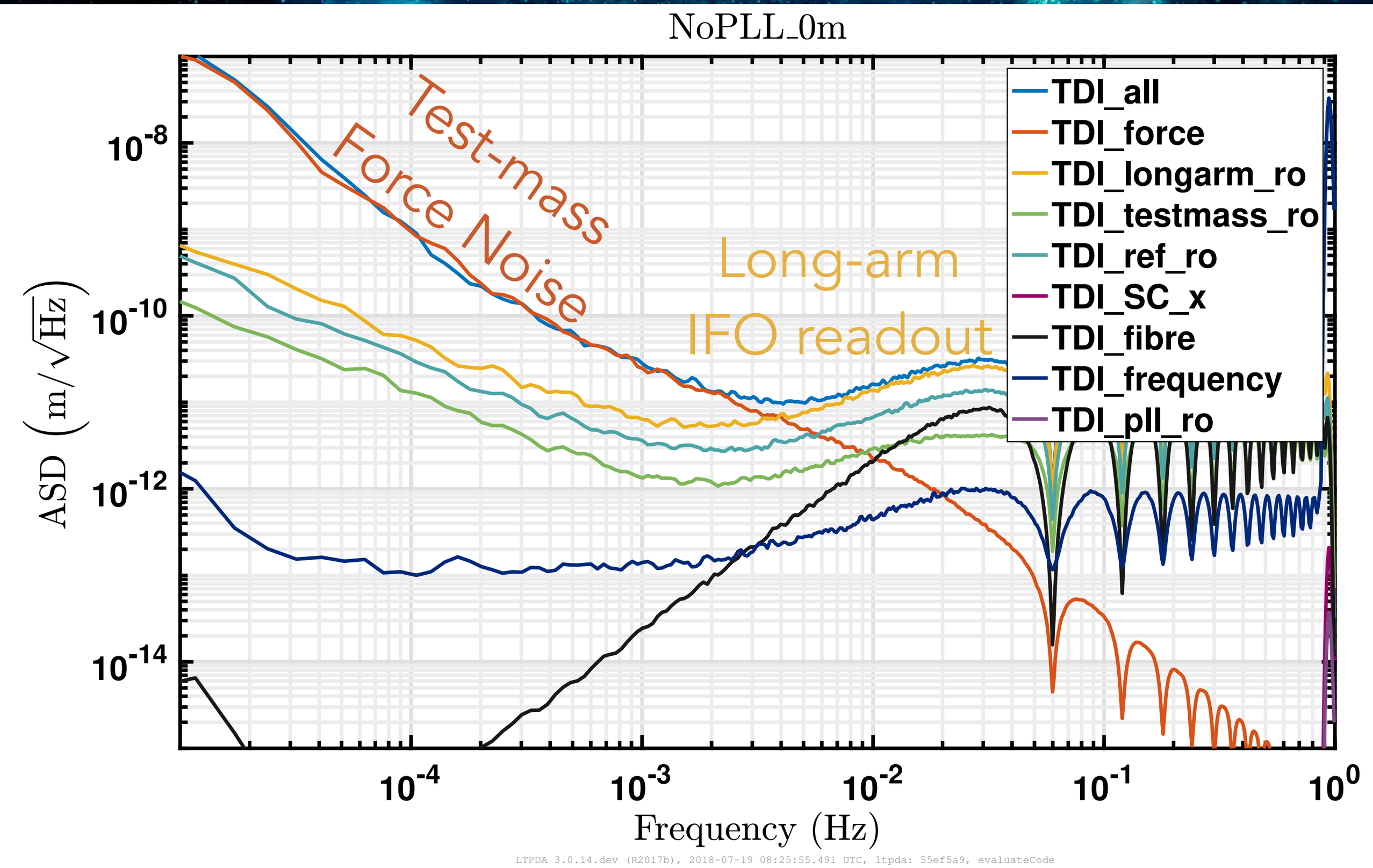
- typical readout noise: $S_{\mathrm{Ns}}^{1/2} = 15\,\mathrm{pm}/\sqrt{\mathrm{Hz}}$

typical level of Brownian force noise on TM

- typical force noise on TM: $S_{\mathrm{force}}^{1/2} = 4\,\mathrm{fN}/\sqrt{\mathrm{Hz}}$

- convert to TM displacement: $S_x^{1/2} = \dfrac{S_{\mathrm{force}}^{1/2}}{M\omega^2}$

Output of X generator using typical numbers



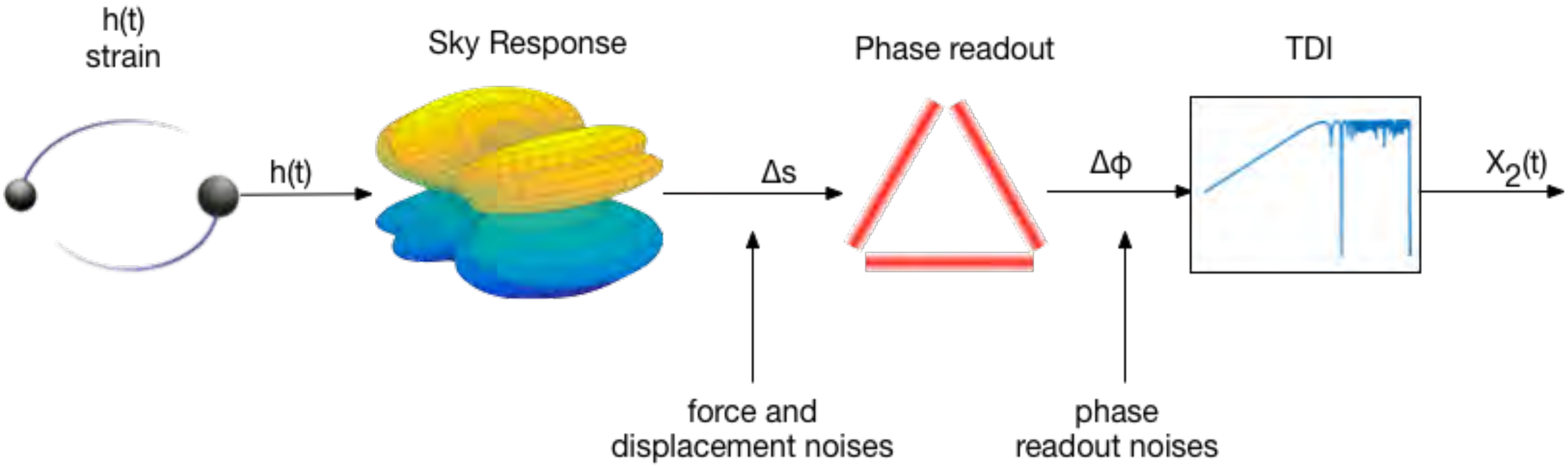
Compute a strain sensitivity curve

- Sensitivity curves are defined as SNR=1 curves averaged over sky position and source polarisation

- TDI transfer function assumed a fixed source in the sky

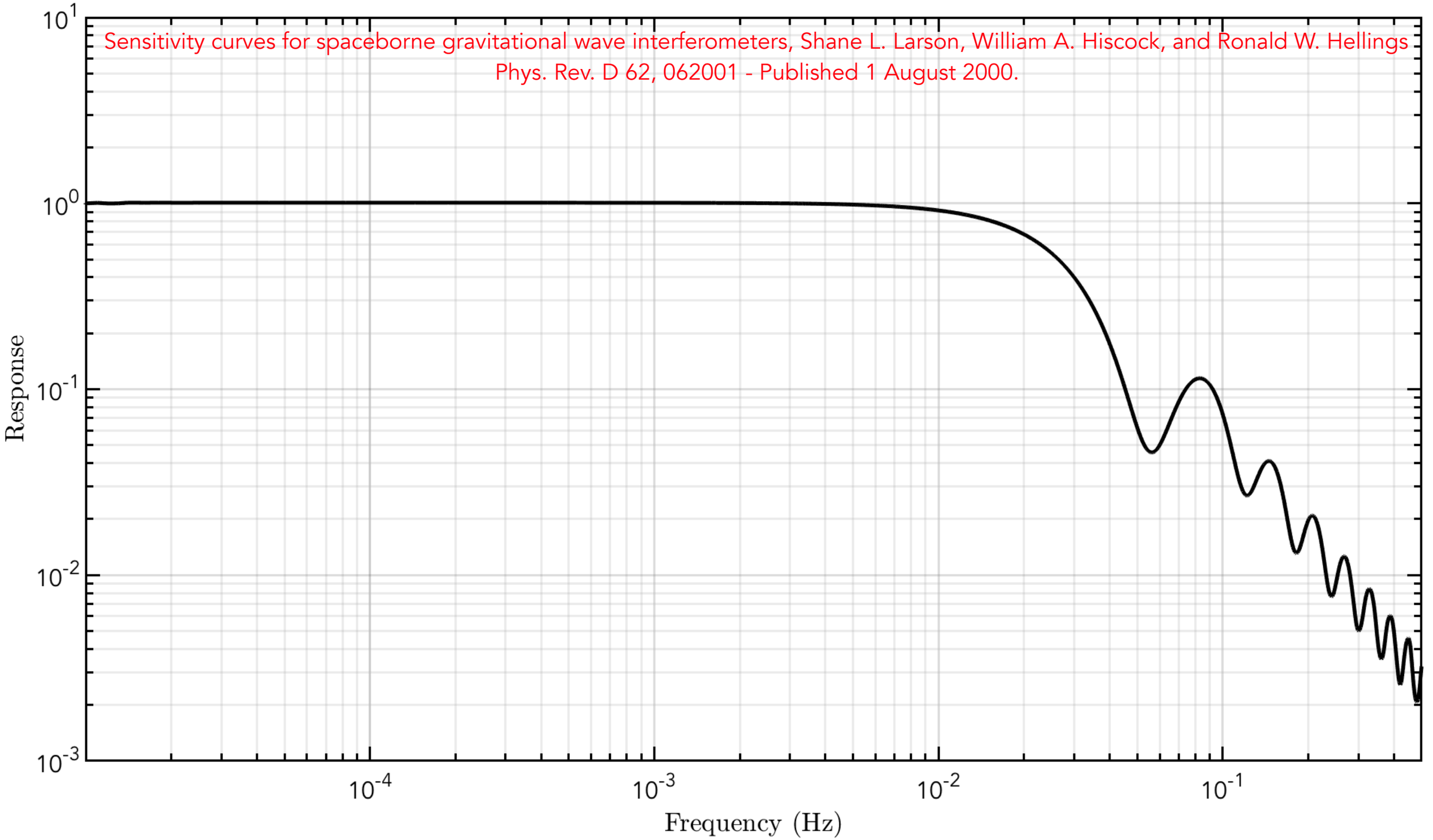
Sensitivity curves for spaceborne gravitational wave interferometers, Shane L. Larson, William A. Hiscock, and Ronald W. Hellings, Phys. Rev. D 62, 062001 - Published 1 August 2000.

Observatory response

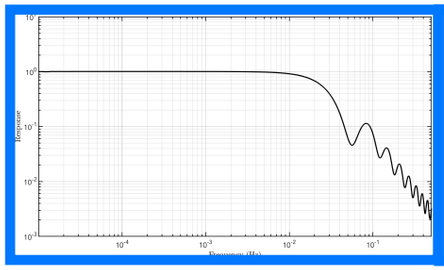


Sky/polarisation averaged observatory response

Sensitivity curves for spaceborne gravitational wave interferometers, Shane L. Larson, William A. Hiscock, and Ronald W. Hellings
Phys. Rev. D 62, 062001 - Published 1 August 2000.

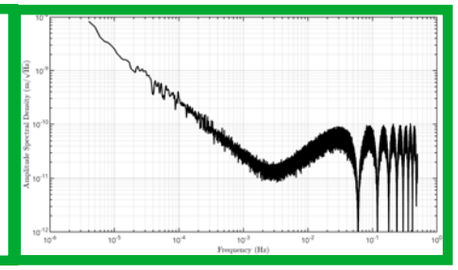


Computing the response



observatory response to GW signal

TDI *X* PSD [m/√Hz]

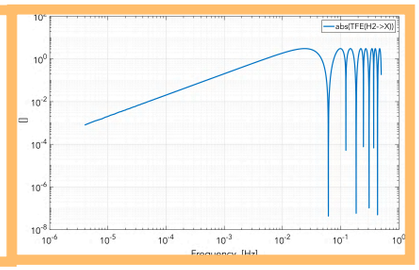


$$S_{\text{strain}} = R(f)\,\frac{5}{\sin 60^\circ}\,\frac{S_X}{T^2_{h\to X}\,L^2}$$

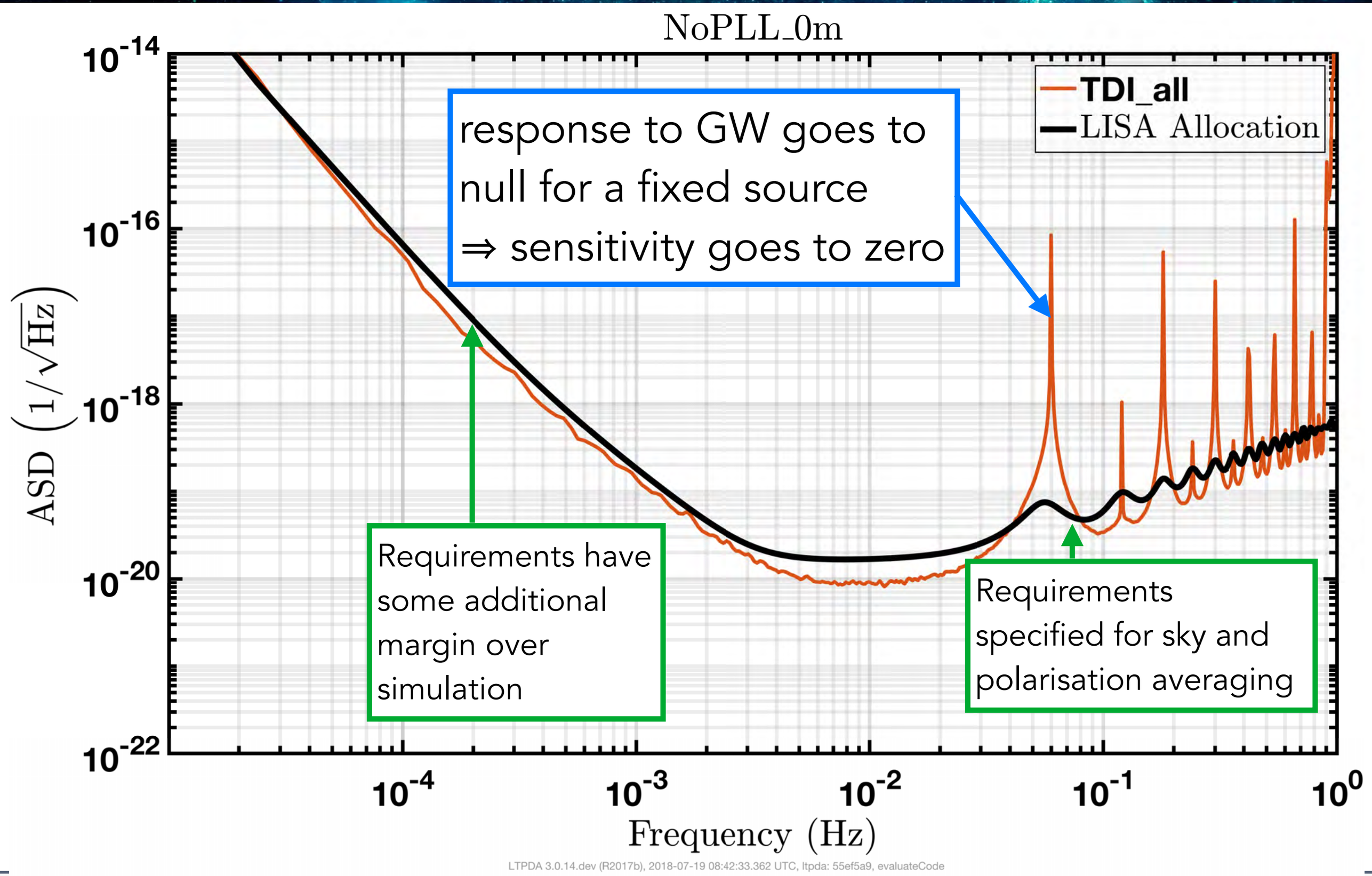
arm-length

arms at 60 degrees
x5 from sky/polarisation averaging

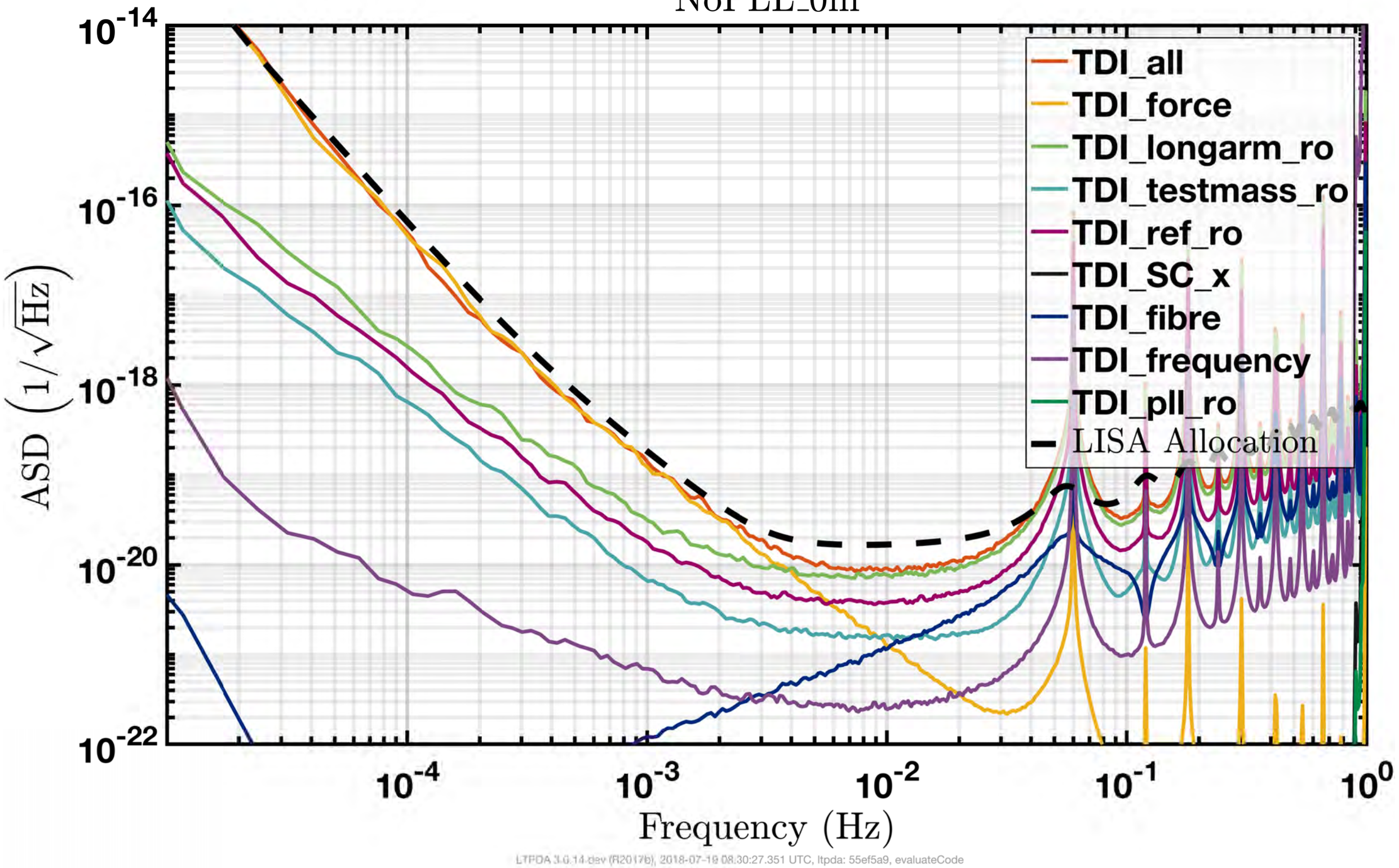
TDI transfer function: GW to *X*



Strain Curve compared to LISA requirements

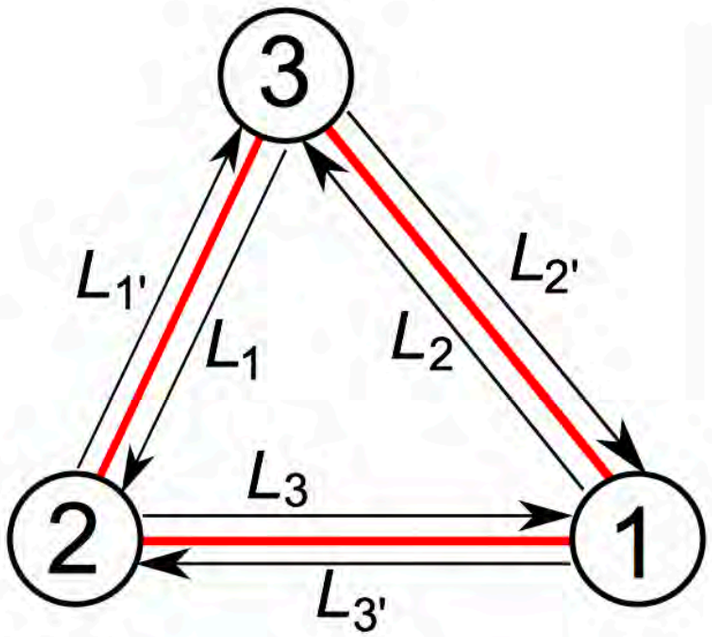


Contributions



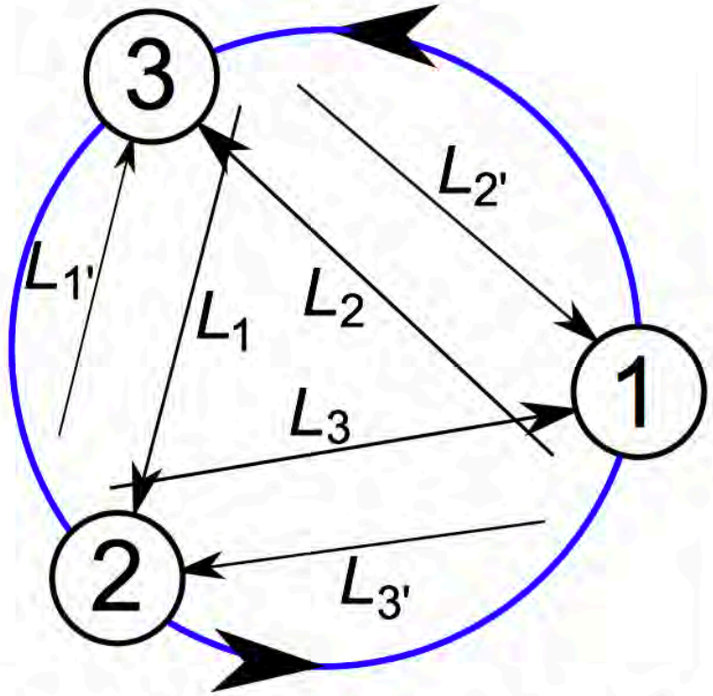
TDI Versions

TDI 1.0



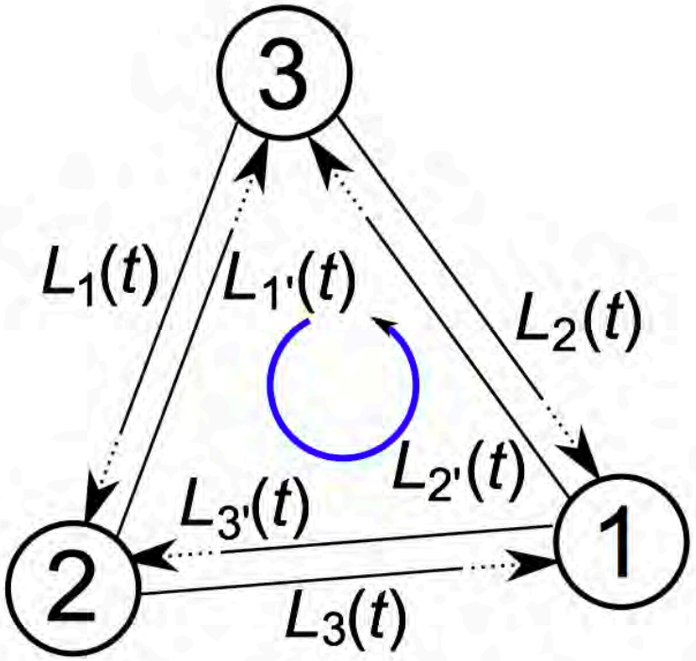
$L_1 = L_{1'},\ L_2 = L_{2'},\ L_3 = L_{3'}$
$L_1 \neq L_2 \neq L_{3'}$

TDI 1.5



$L_1 \neq L_{1'} \neq L_2 \neq L_{2'} \neq L_3 \neq L_{3'}$

TDI 2.0



$L_1 \neq L_{1'} \neq L_2 \neq L_{2'} \neq L_3 \neq L_{3'}$
$L_1 = L_1(t),\ L_{1'} = L_{1'}(t),\ L_2 = L_2(t),$
$L_{2'} = L_{2'}(t),\ L_3 = L_3(t),\ L_{3'} = L_{3'}(t)$

You thought that was bad?

- Full solution has to deal with
  - clock-noise on sampled data streams
  - sign switching of beats to deal with doppler shifts
  - full time-varying, unequal arm lengths (delays)
  - non-reciprocal back-link fibre
  - lasers with slightly different wavelengths

$$
\begin{aligned}
X_1 \sim\ & H_{1'} + H_{3;2'} + H_{1;22'} + H_{2';322'} + H_{1;3'322'} + H_{2';33'322'} \\
& + H_{1';3'33'322'} + H_{3;2'3'33'322'} - H_1 - H_{2';3} - H_{1';3'3} - H_{3;2'3'3} \\
& - H_{1';22'3'3} - H_{3;2'22'3'3} - H_{1;22'22'3'3} - H_{2';322'22'3'3} \\
& + \frac{\mu_{1'} - \mu_1}{2} + \frac{\mu_{1;3'3} - \mu_{1';3'3}}{2} + \frac{\mu_{1;22'} - \mu_{1';22'}}{2} \\
& + \frac{\mu_{1';22'22'3'3} - \mu_{1;22'22'3'3}}{2} + \frac{\mu_{1';3'33'322'} - \mu_{1;3'33'322'}}{2} \\
& + \frac{\mu_{1;22'3'33'322'} - \mu_{1';22'3'33'322'}}{2} + \mathcal{O}\left(\frac{\dot{L}^2}{c^2}\right), \qquad (4.17)
\end{aligned}
$$

Otto, Time-Delay Interferometry Simulations for LISA, PhD Thesis 2015