# Mathematics and the prints of M.C. Escher

Joe Romano

Les Houches Summer School

23 July 2018

# Possible topics

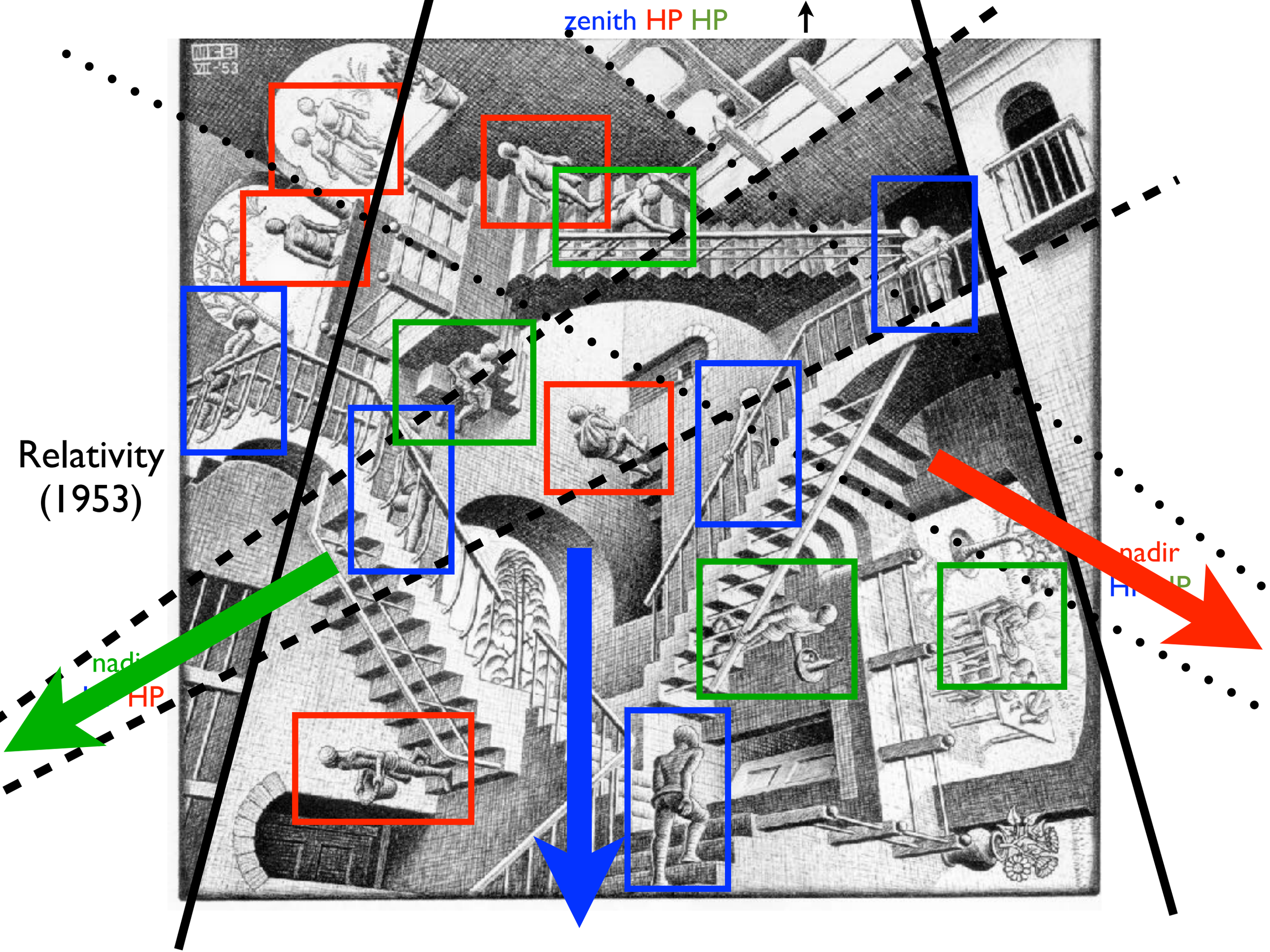
- projective geometry
- non-Euclidean geometry
- topology & knots
- duality & complementarity
- symmetry (periodic tilings)
- representations of infinity
- ambiguous perspective
- impossible objects
- …

# Plan of talk

1. Escher - brief biography (connection to R. Penrose)
2. Symmetry transformations - periodic tilings of the plane
3. Periodic tilings using regular polygons (non-Euclidean)

# I. Escher the person

- Maurits Cornelis Escher (1898-1972)
- Dutch graphic artist (woodcuts, lithographs)
- Inspired by visits to the Alhambra in Granada, Spain in 1922 and 1936
- From 1937, the topic of periodic tilings of the plane was his "richest source of inspiration"
- He was not a mathematician but thought like a mathematician (corresponded with Coxeter and Penrose)


zenith HP HP
Relativity
(1953)
nadir
HP

# IMPOSSIBLE OBJECTS: A SPECIAL TYPE OF VISUAL ILLUSION

By L. S. PENROSE and R. PENROSE

(*University College, London, and Bedford College, London*)

British Journal of Psychology, 49:1 (Feb 1958), p.31

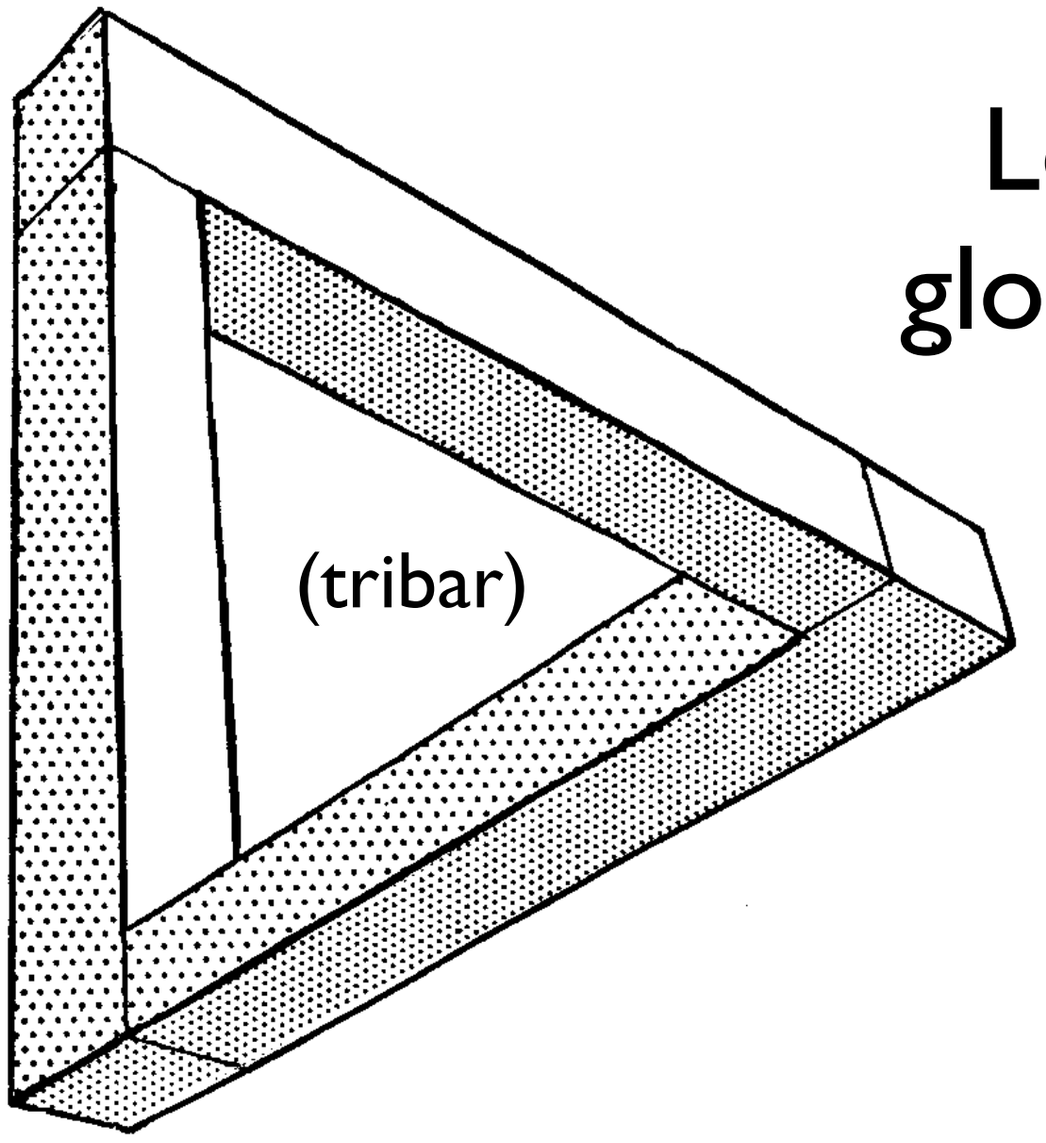
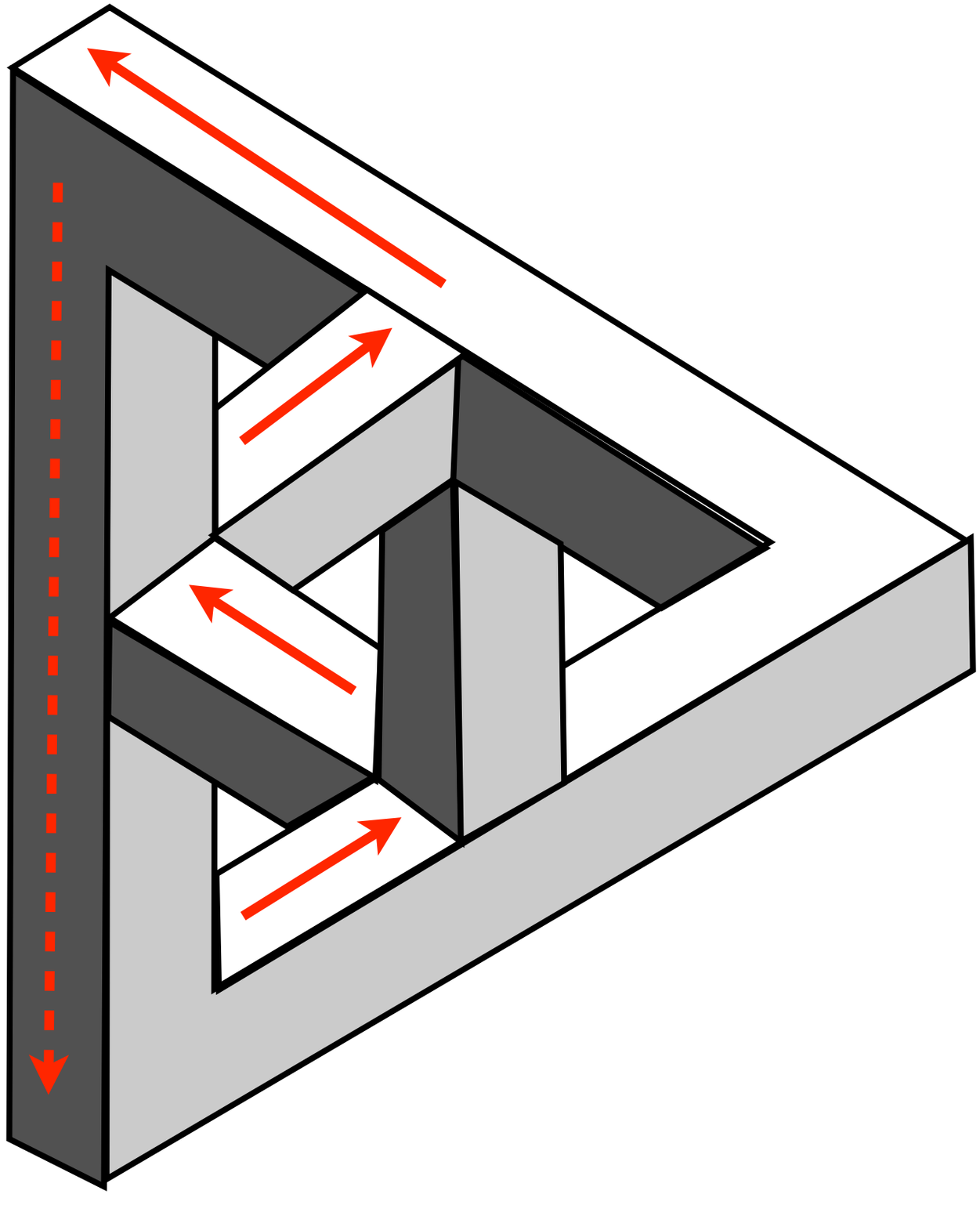
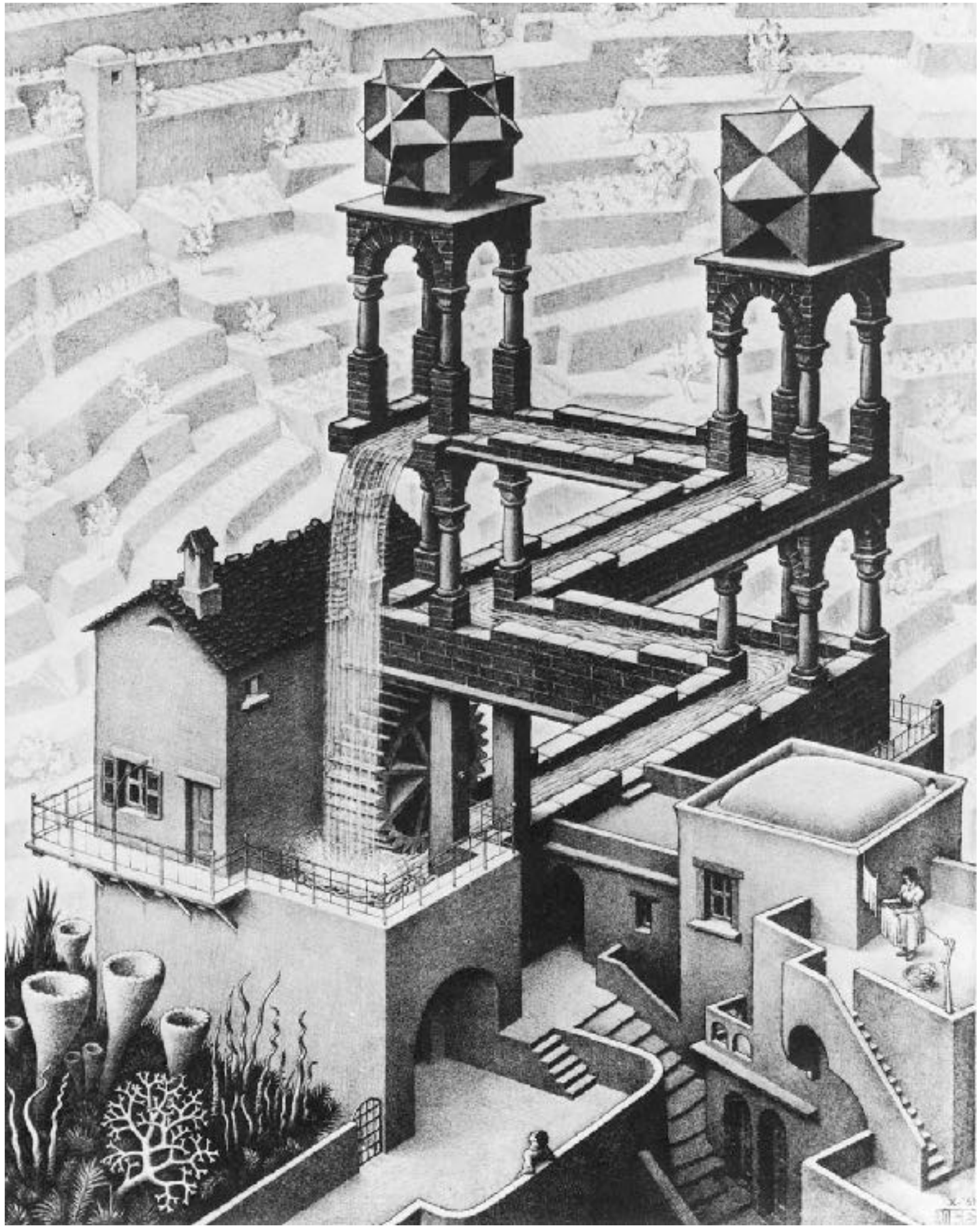



Fig. 1. Perspective drawing of impossible structure.

Local possibility, global impossibility!



Waterfall
(1961)



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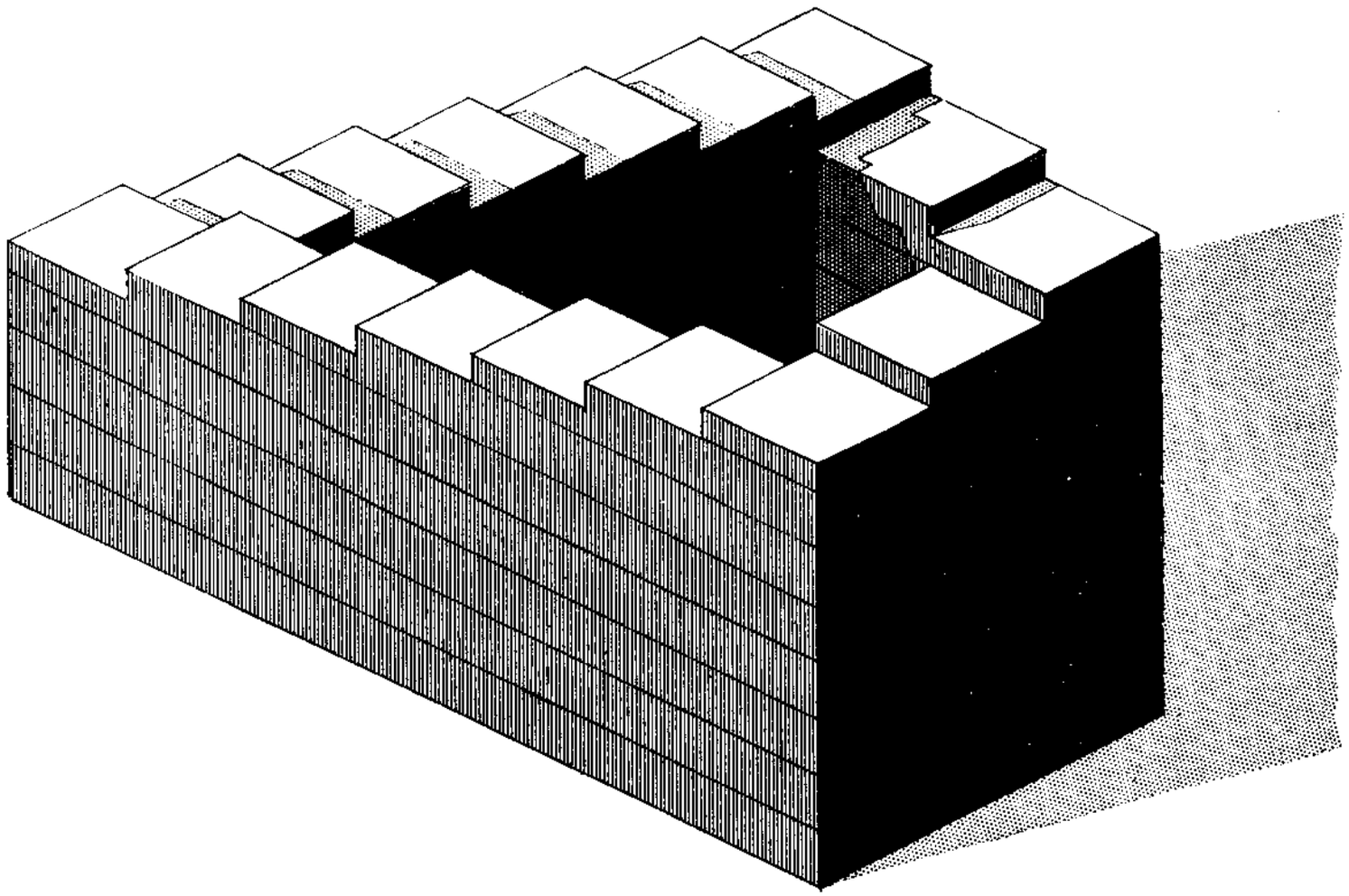
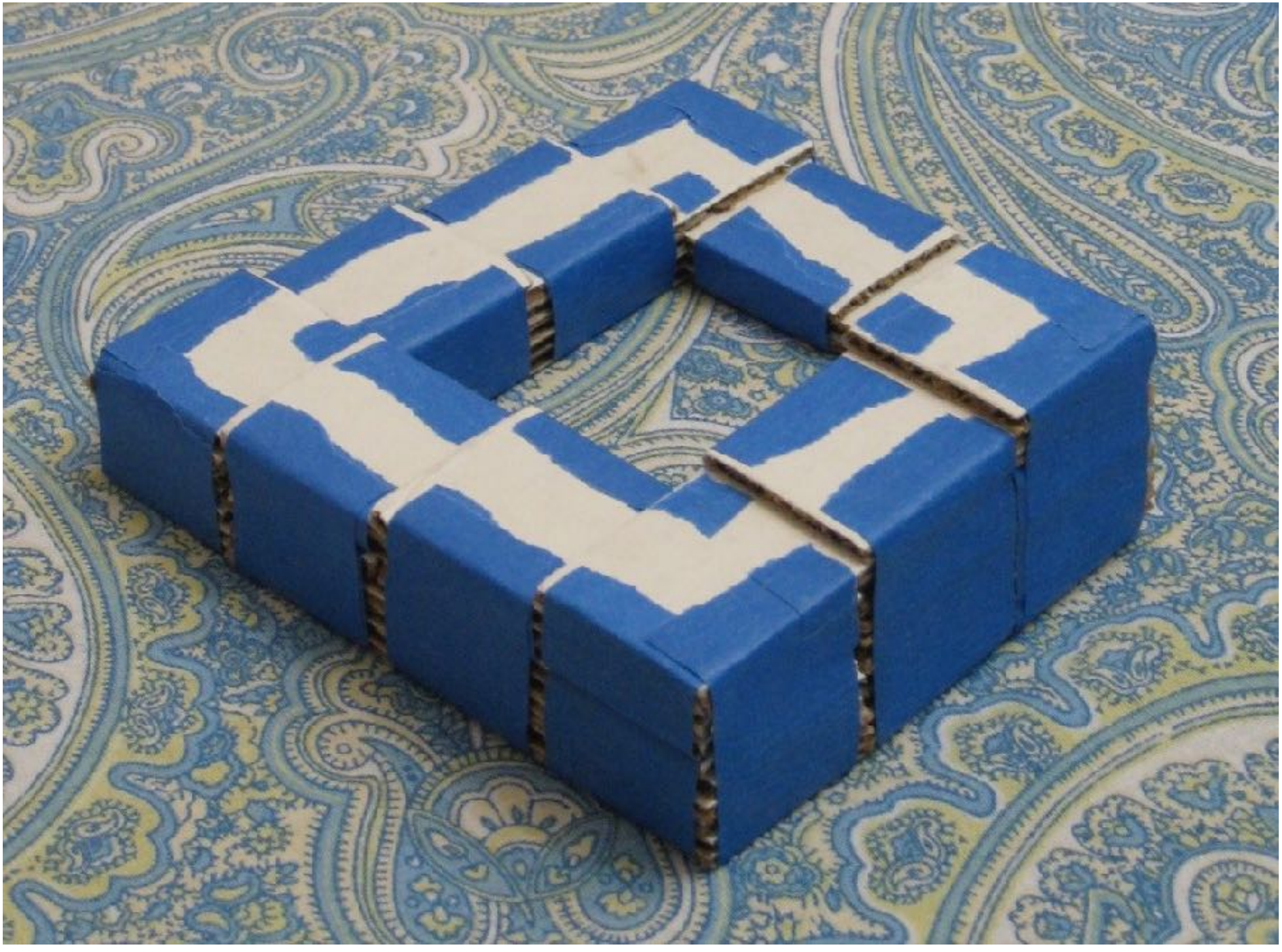
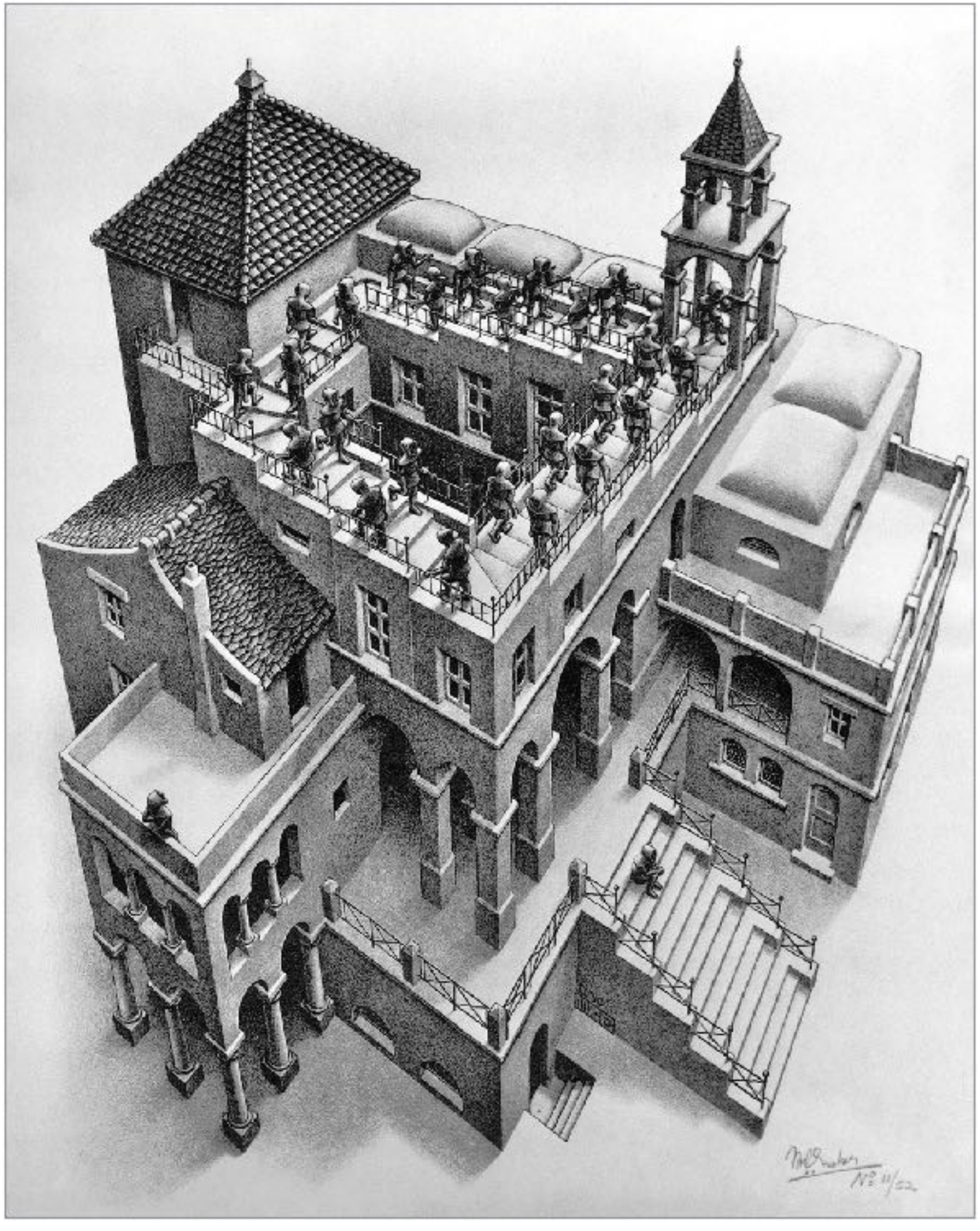


**Fig. 3.** Continuous flight of steps: shadowed drawing.



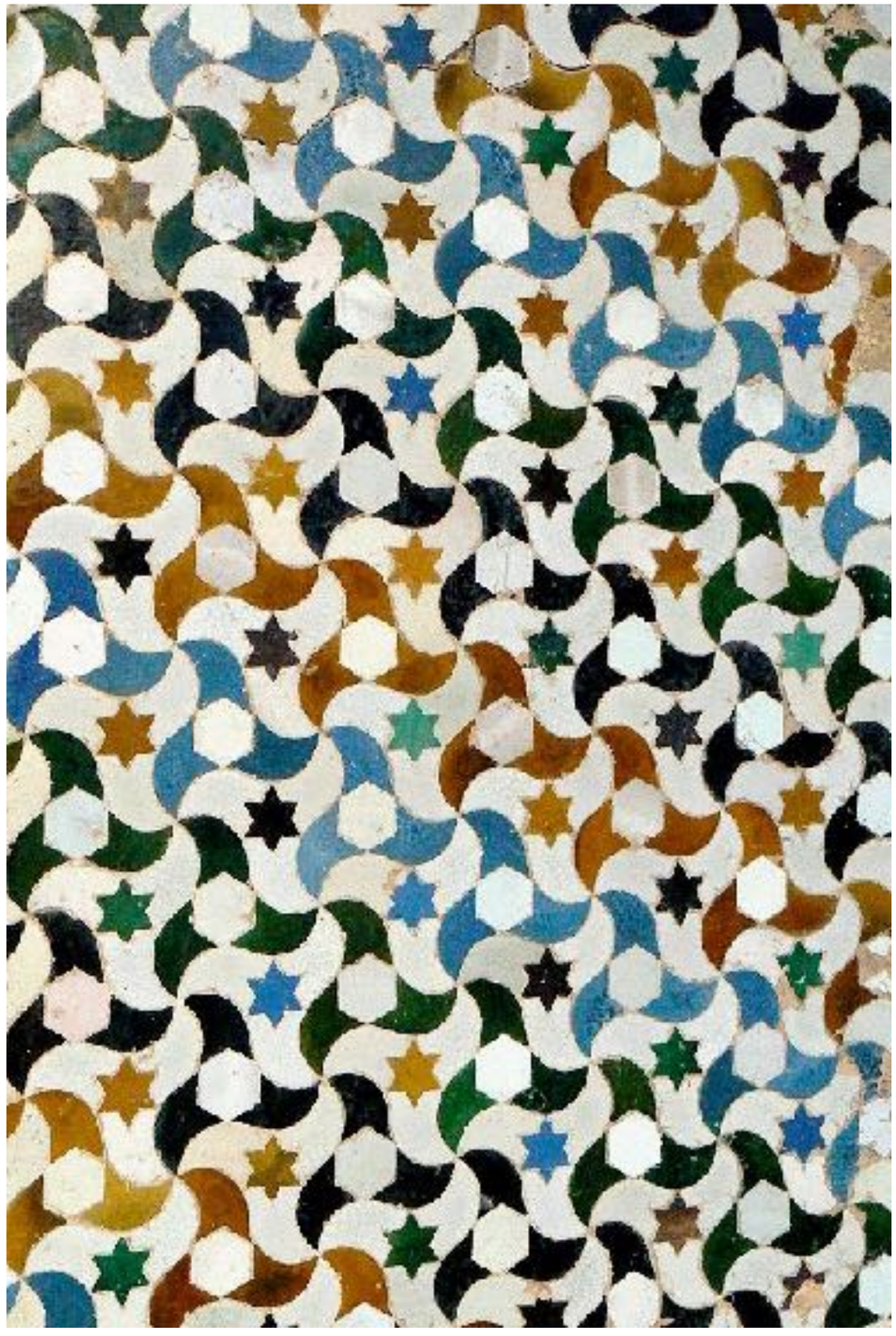
Ascending and Descending (1960)



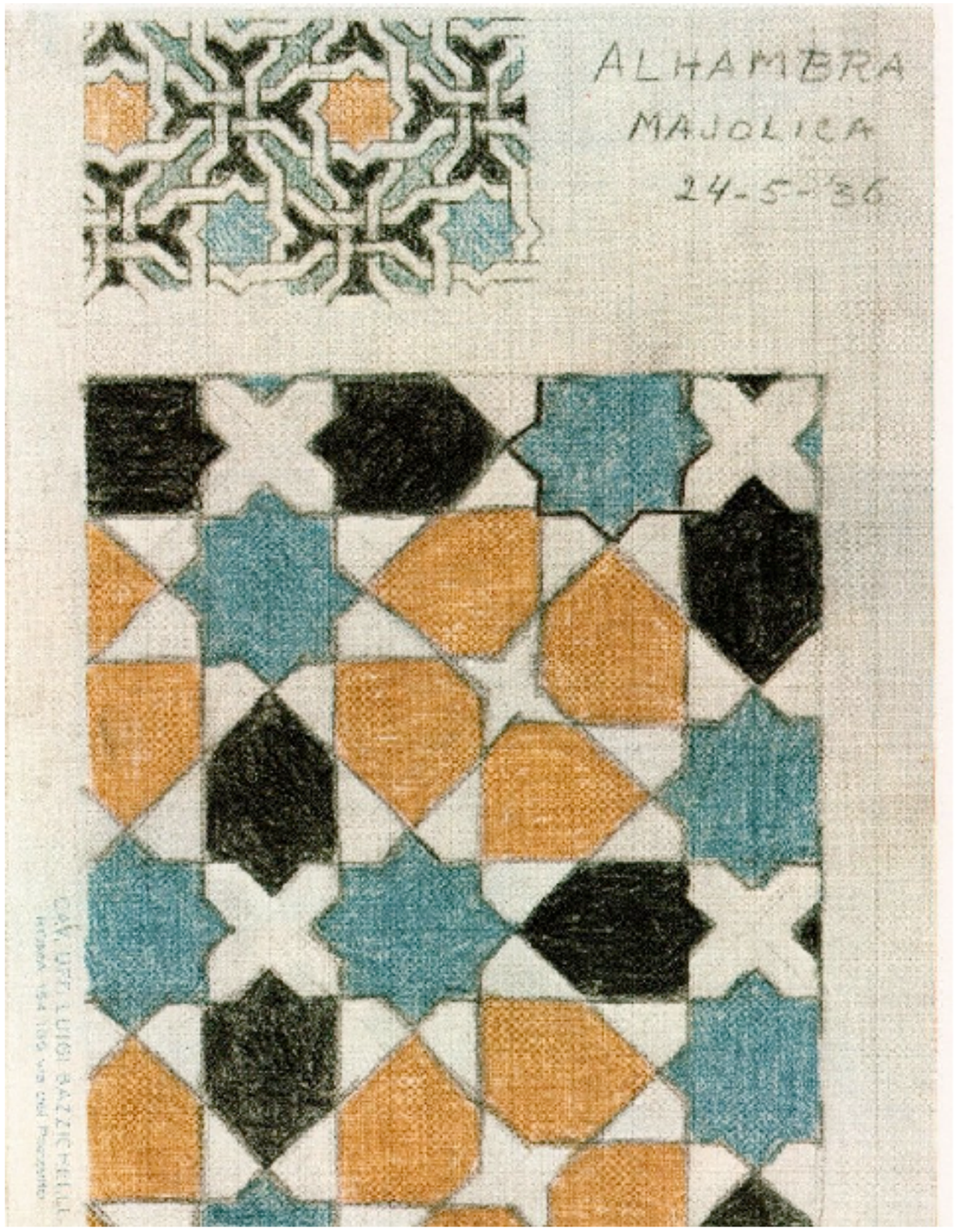
# 2. Symmetry transformations

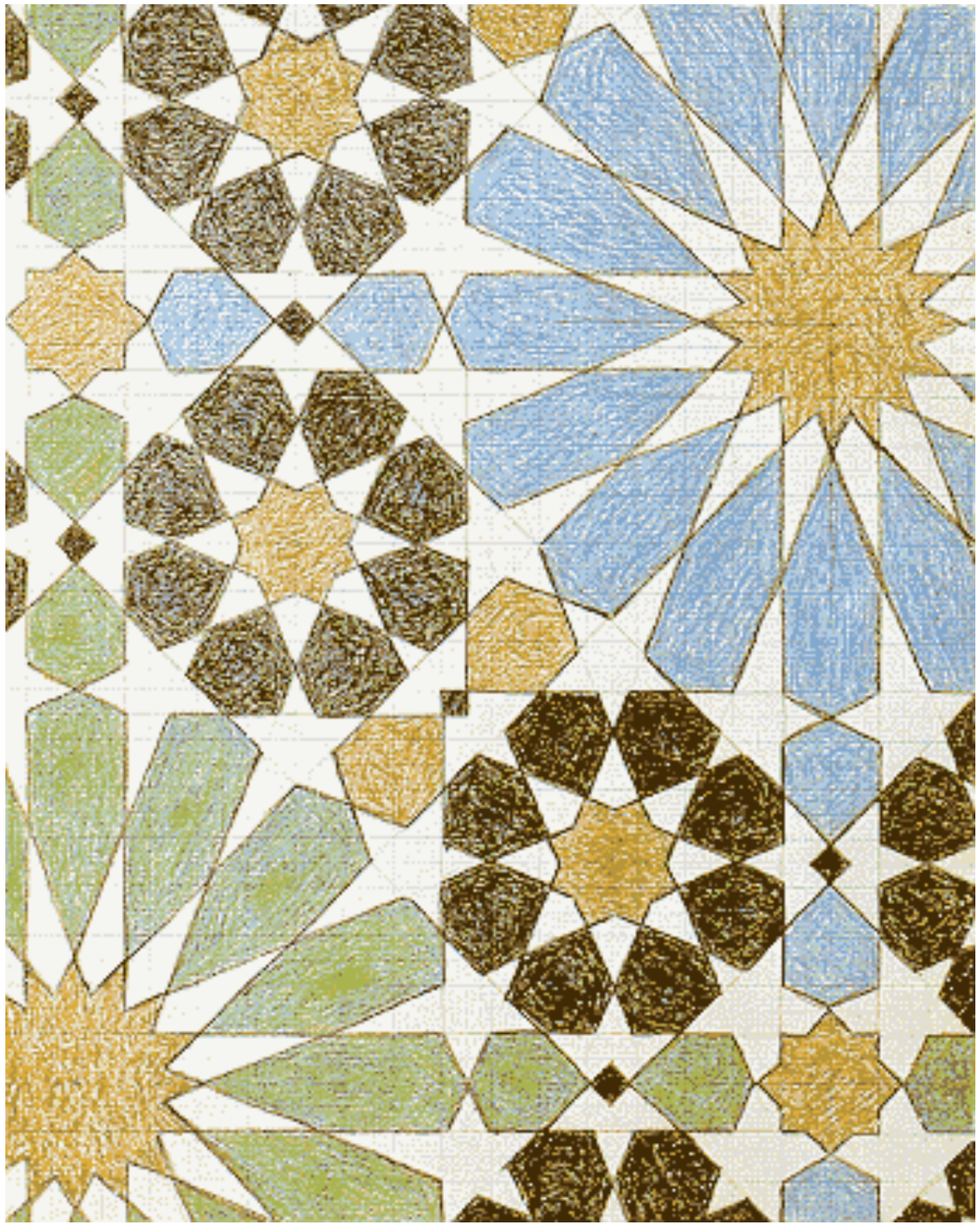
# The Alhambra (Granada, Spain)

fortress (889)
palace (1333)



# Escher's sketches



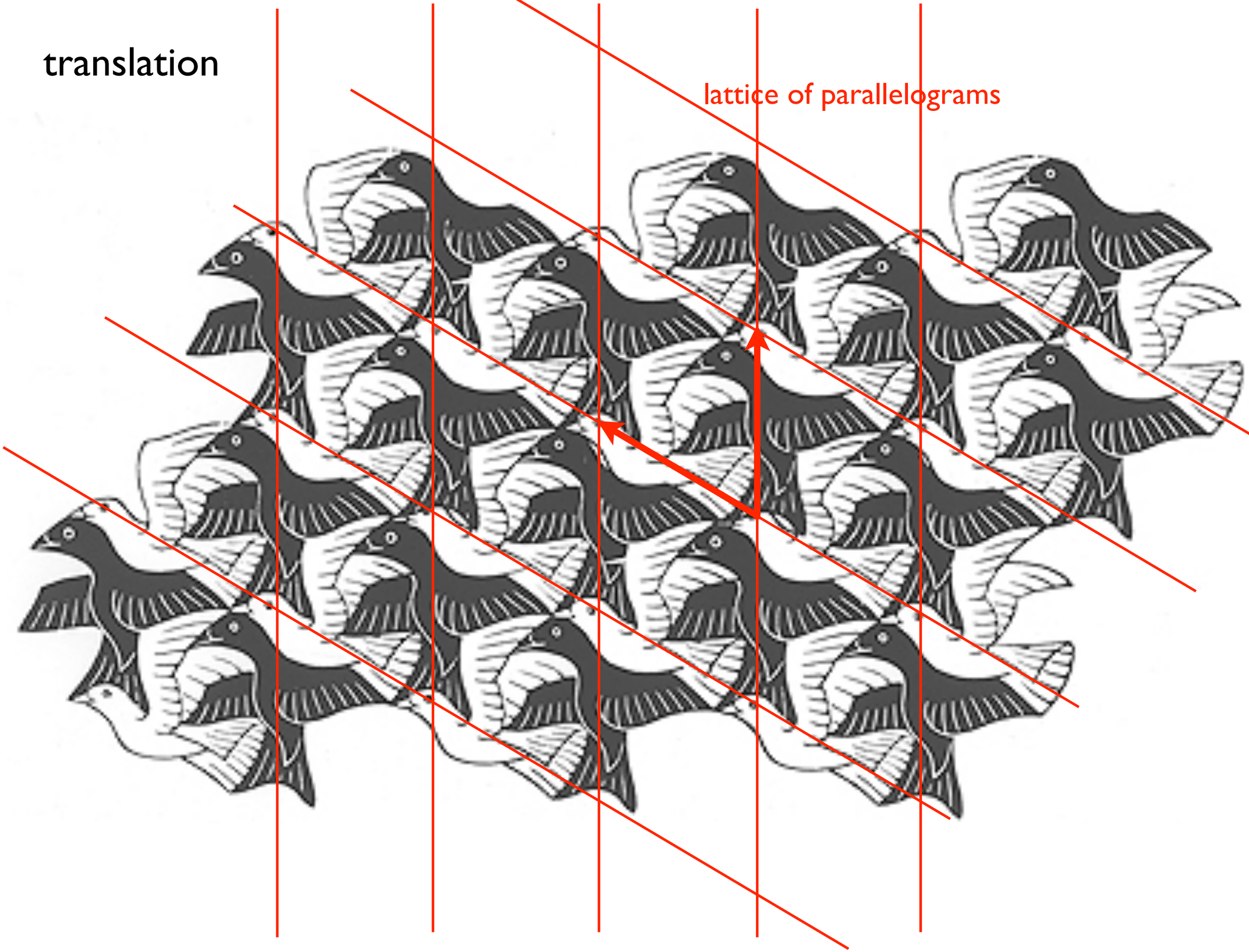



# Some observations

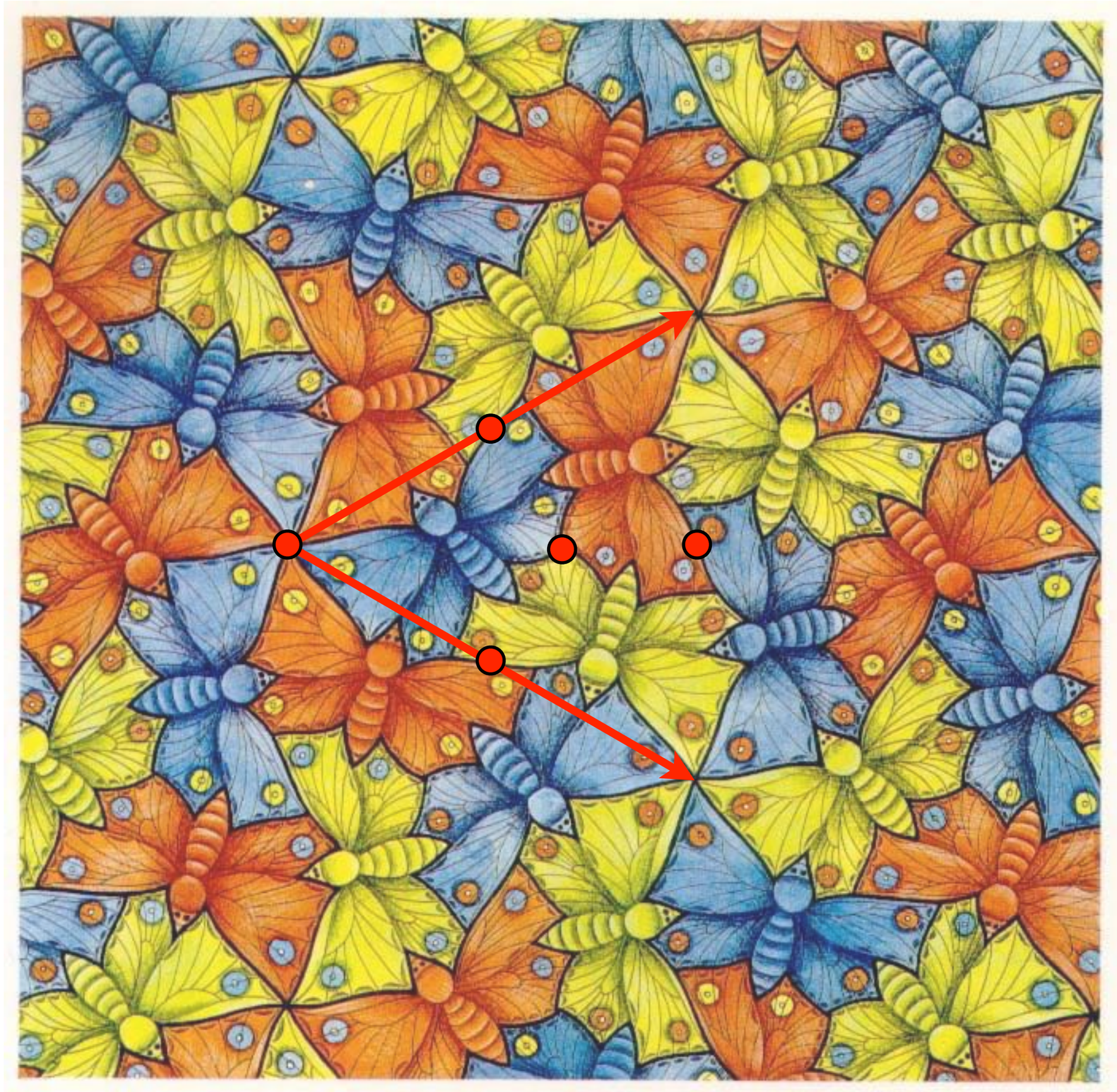
- Alhambra wall tilings are periodic tilings of the plane (complete covering of a flat 2-d surface using repeated congruent figures)
- Instead of geometric shapes, Escher wanted to use recognizable figures (e.g., birds, fish, lizards, ...), forbidden by religion of Moors
- Congruent figures (same size & shape) limits allowed transformations (translations, rotations, reflections, glide reflections)
- Symmetric with respect to a particular transformation means it is unchanged after applying that transformation

translation

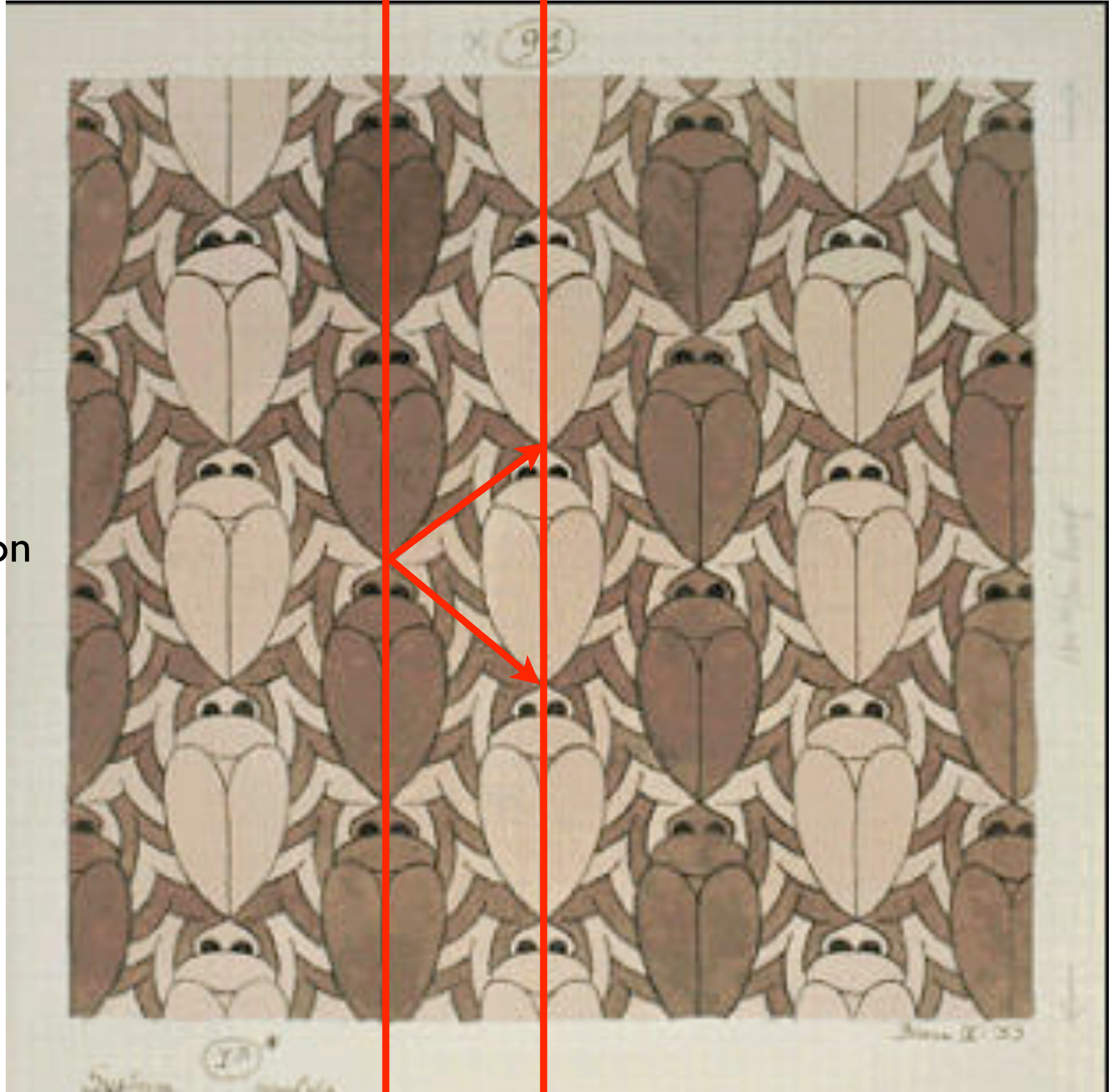
lattice of parallelograms



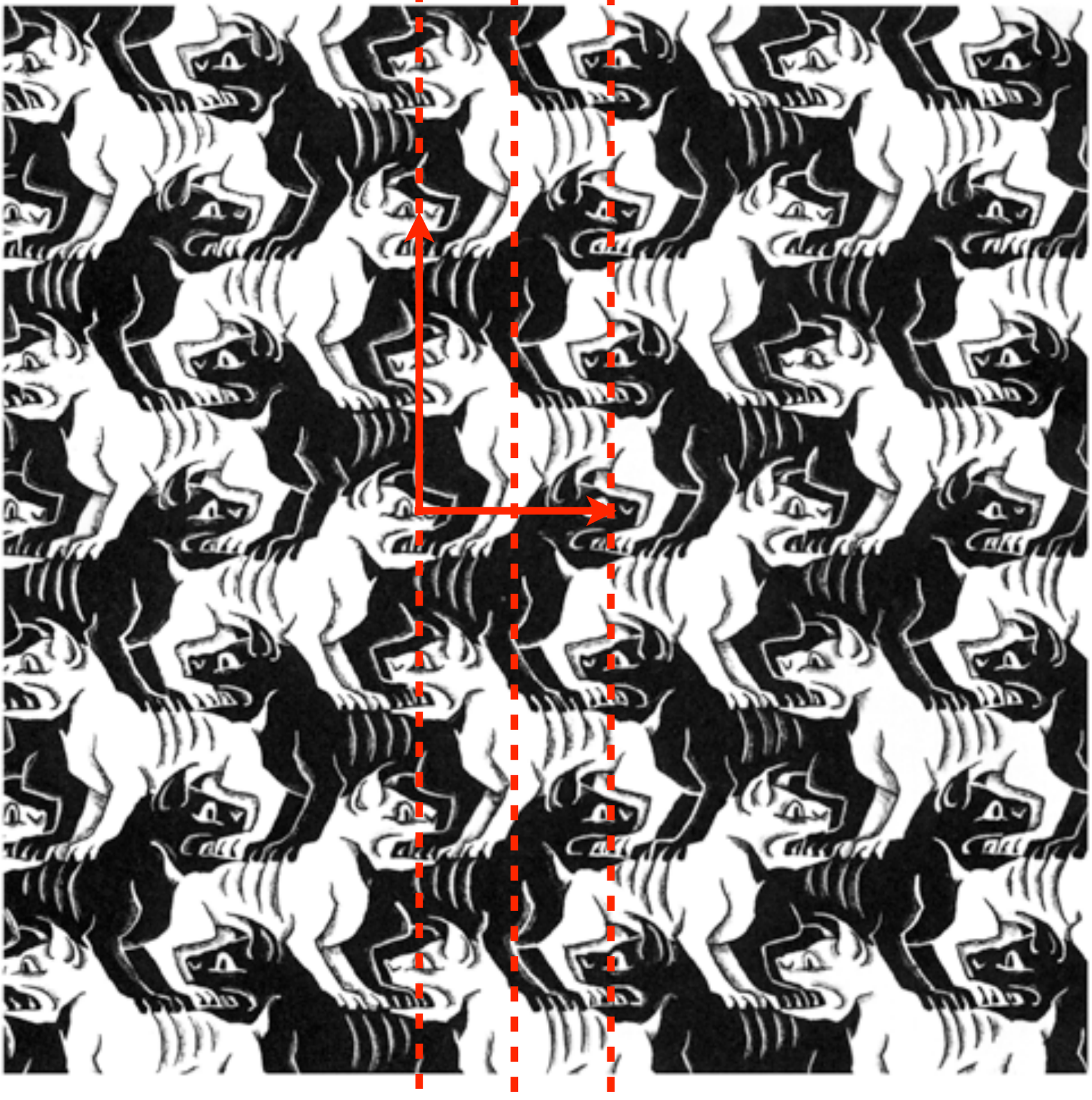
rotation
(2, 3, 6)



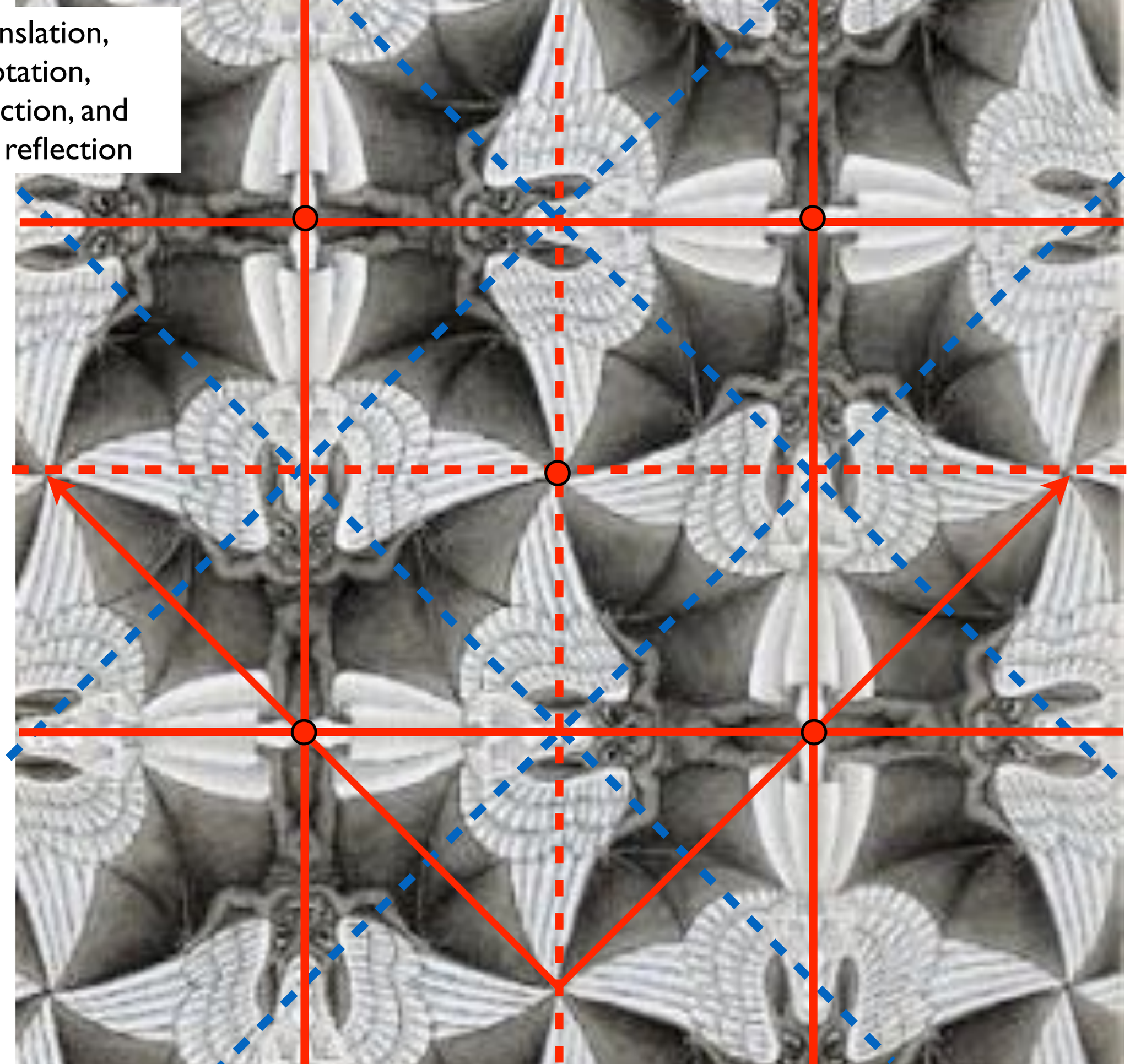
reflection



glide
reflection



translation,
rotation,
reflection, and
glide reflection



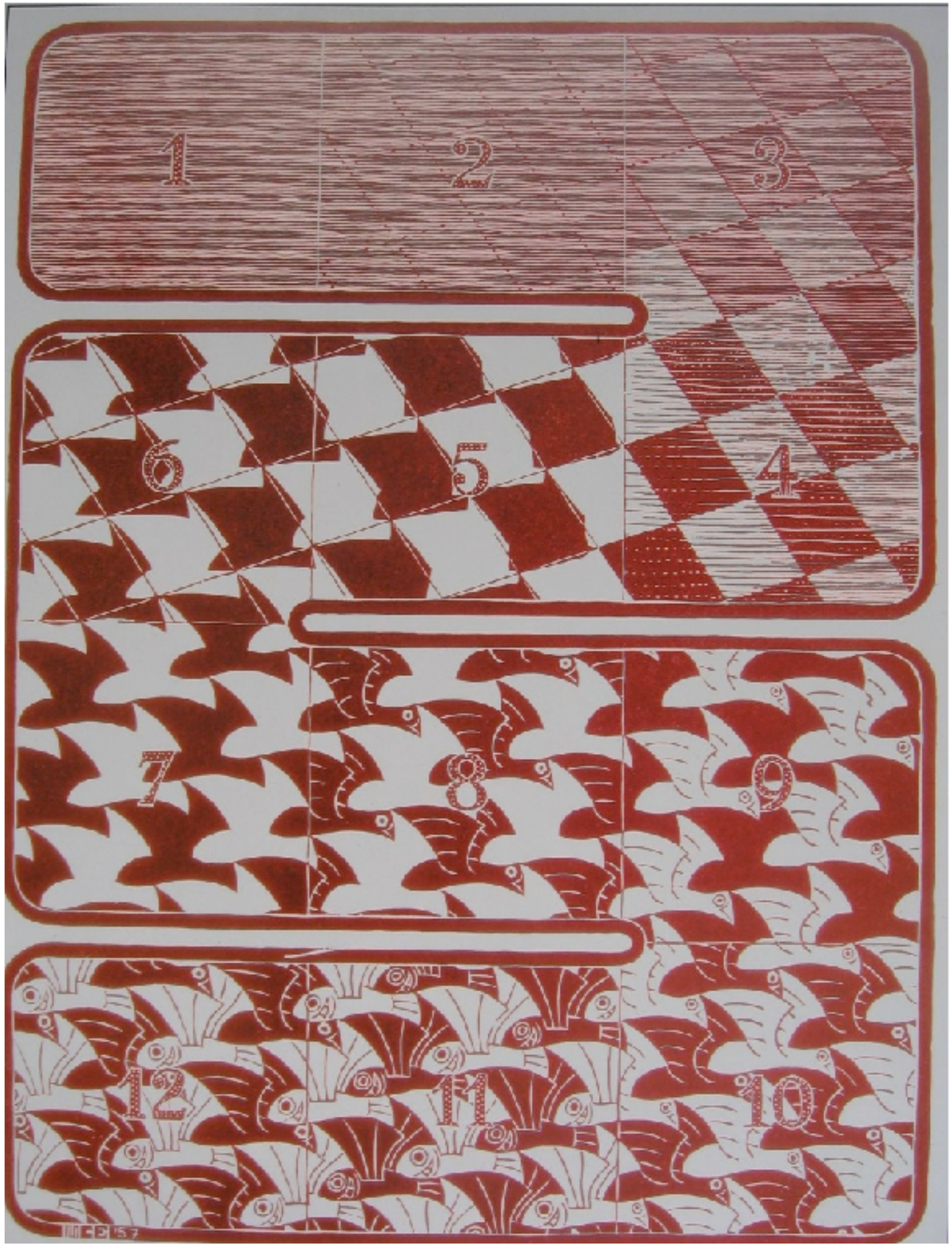
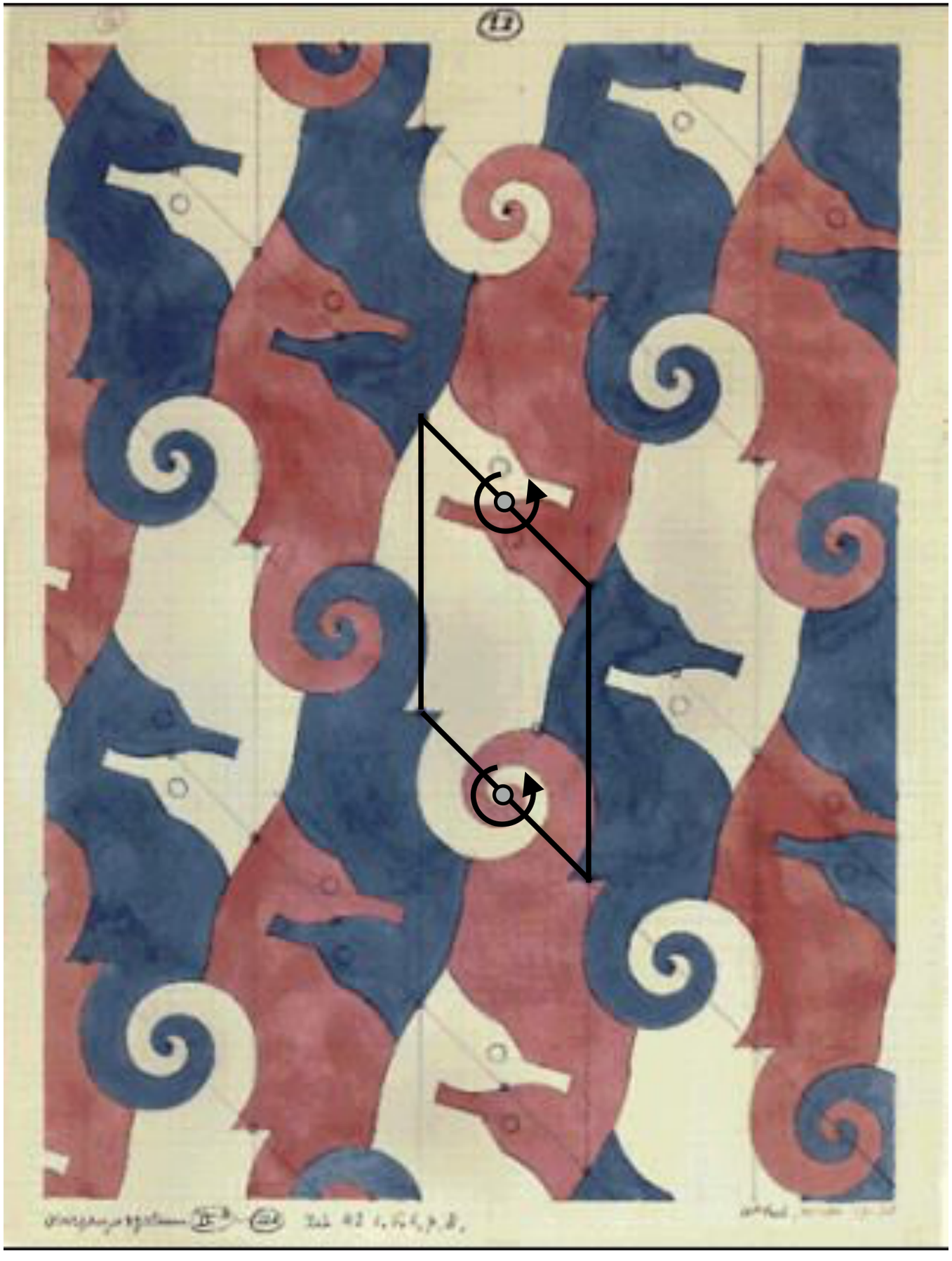



Figure 69
(Plane Tessellations)



28 different ways of constructing asymmetric tiles (Heesch group)

# 3. Periodic tilings using regular polygons

# Q: How many ways can you tile 2-d flat space using regular polygons?

A. Zero

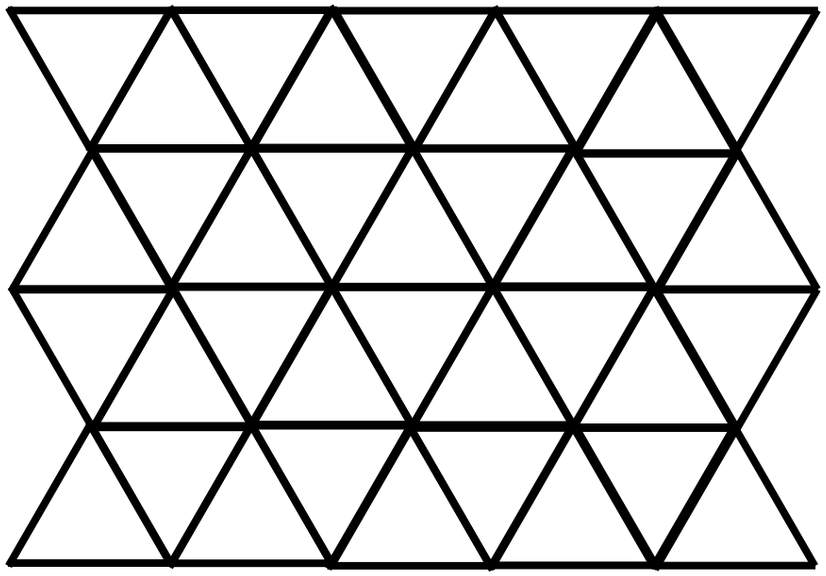
B. Three

C. Five

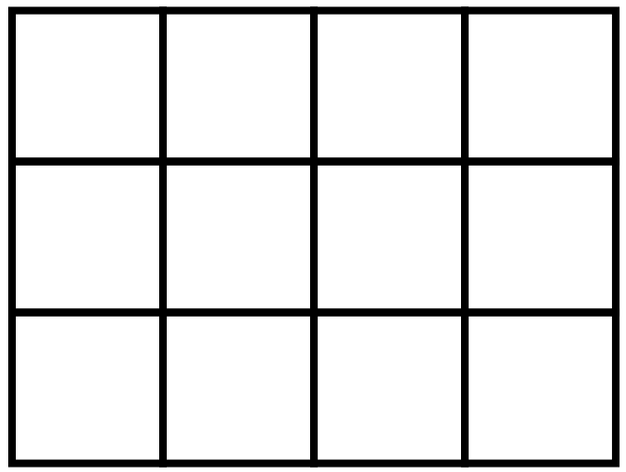
D. Infinity

Answer: Three (equilateral triangles, squares, or hexagons)

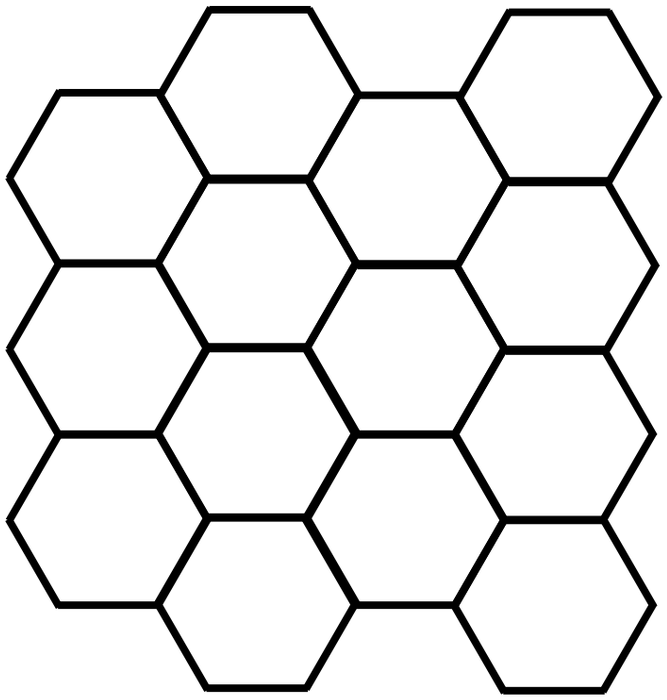
# Tilings of 2-d flat space (i.e., plane) using regular polygons



{p,q} = {3,6}



{4,4}



{6,3}

Q: But how do you prove that these are the only three?

A: Sum of the angles around each vertex = 360°

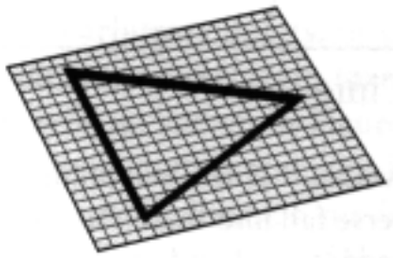
Opening angle of a regular p-gon = (p-2)∗180°/p

q∗ (p-2)∗180° / p = 360°

Tiling condition for q p-gons meeting at a vertex: 1/p + 1/q = 1/2

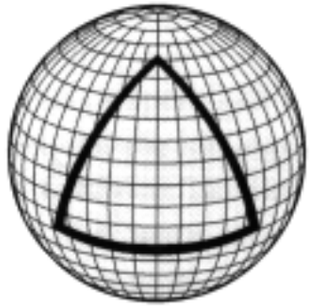
# Extension to 2-d sphere and hyperbolic space

Straight lines in a curved space are the shortest distance curves



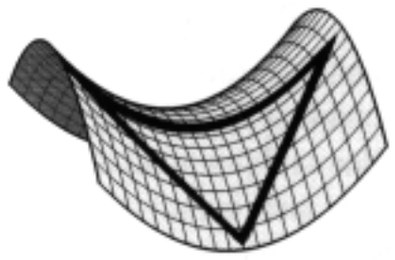
flat (Euclidean)

1. Parallel lines never intersect
2. Sum of angles of a triangle =180°
3. $C = 2\pi r$



positively curved
(spherical, elliptic)

1. Initially parallel lines intersect
2. Sum of angles of a triangle >180°
3. $C < 2\pi r$



negatively curved
(hyperbolic)

1. Initially parallel lines diverge
2. Sum of angles of a triangle <180°
3. $C > 2\pi r$

# Q: How many ways can you tile a 2-d sphere using regular polygons?

A. Zero

B. Three

C. Five

D. Infinity

Answer: 5 or infinity

Proof: Tiling condition for 2-d sphere is $1/p + 1/q > 1/2$

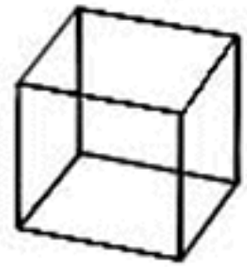
$\{p,q\} = \{3,3\}, \{4,3\}, \{3,4\}, \{5,3\}, \{3,5\}, \{2,n\}, \{n,2\}$

# Tilings of the sphere using regular polygons: 1/p + 1/q > 1/2

{3,3}

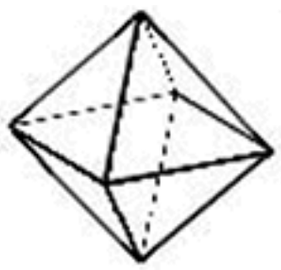
tetrahedron

{4,3}



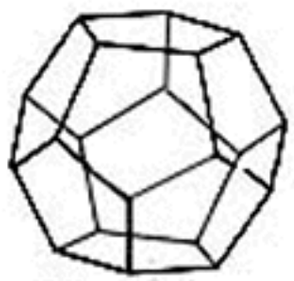
cube

{3,4}



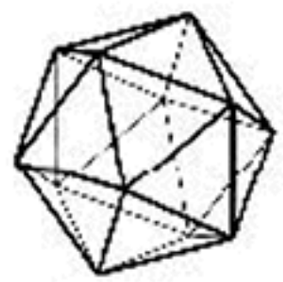
octahedron

{5,3}



dodecahedron

{3,5}



icosahedron

## Degenerate tessellations

{2,n}

(biangles)

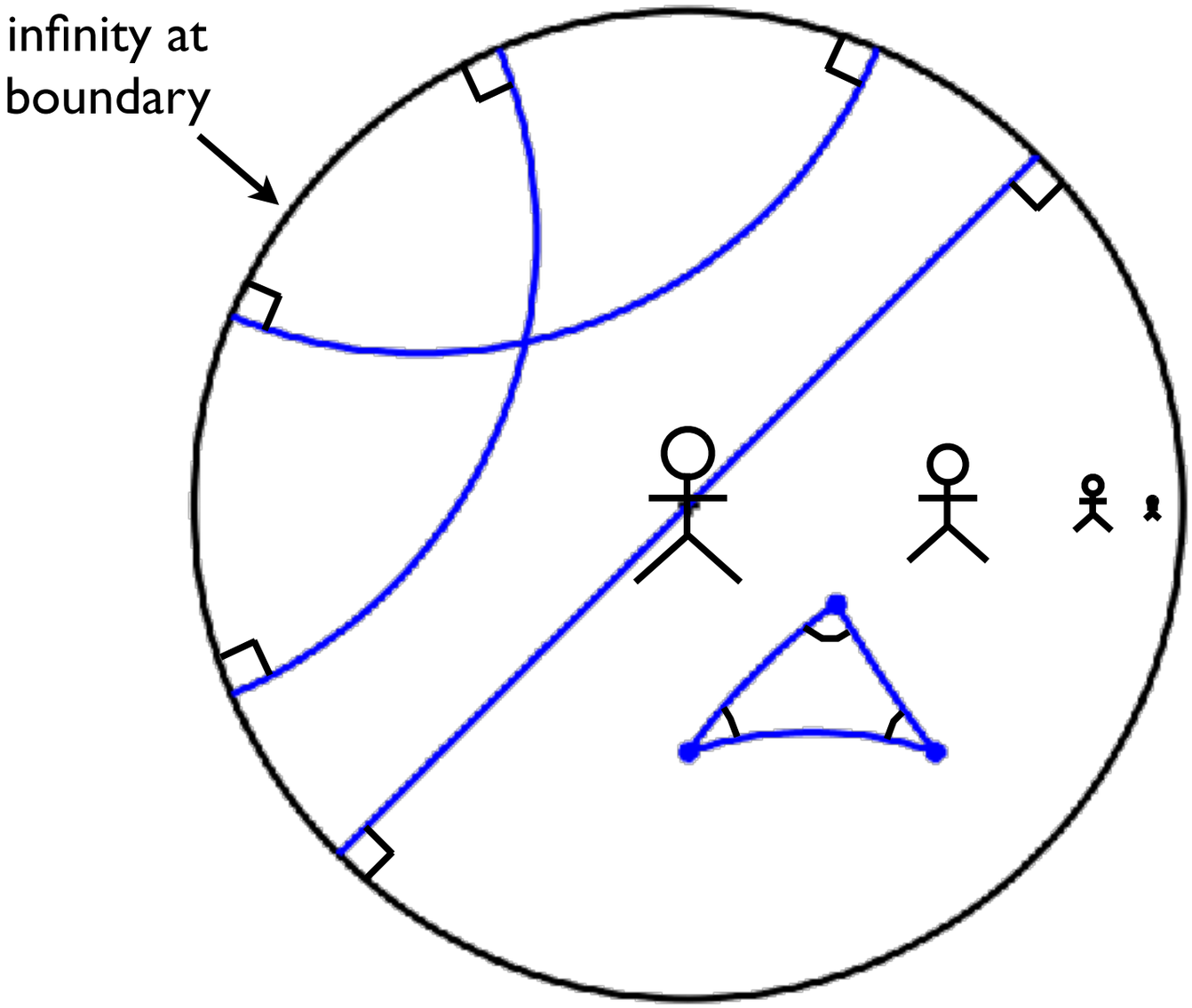
{n,2}

(from EscherMath)

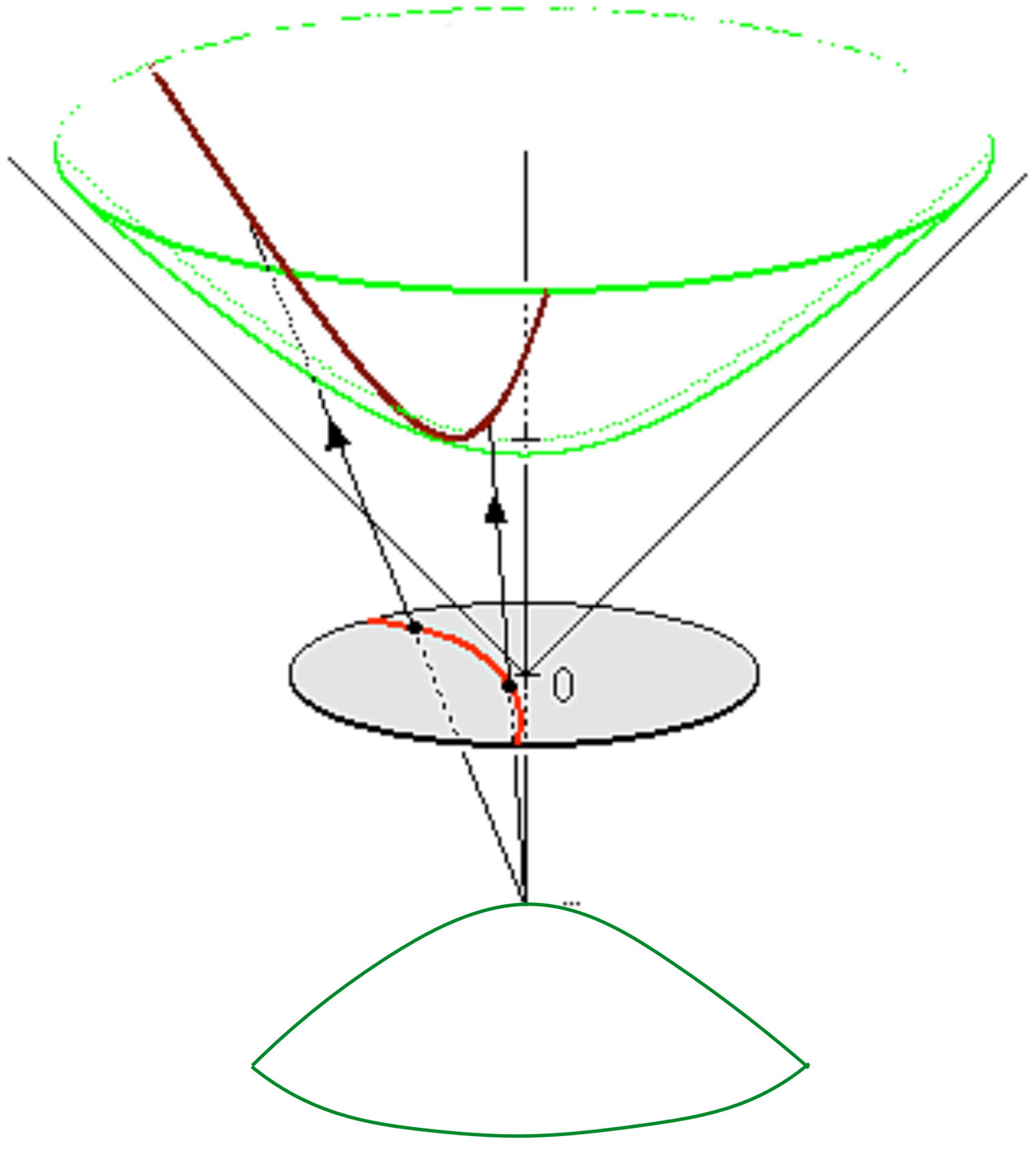
Sphere with Angels and Devils (1942)

(regular quadrilaterals meeting 3 at a vertex)

# 2-d hyperbolic space with constant negative curvature (Poincaré disk model)




- finite representation of an infinite space with const negative curvature
- straight lines are arcs of circles that intersect the boundary at right angles
- sum of the angles of a triangle < 180°
- conformal representation (angles preserved)
- distances are distorted as you move toward the boundary: $1-(\rho/R)^2$




(from wikipedia)

# Q: How many ways can you tile 2-d hyperbolic space using regular polygons?

A. Zero

B. Three

C. Five

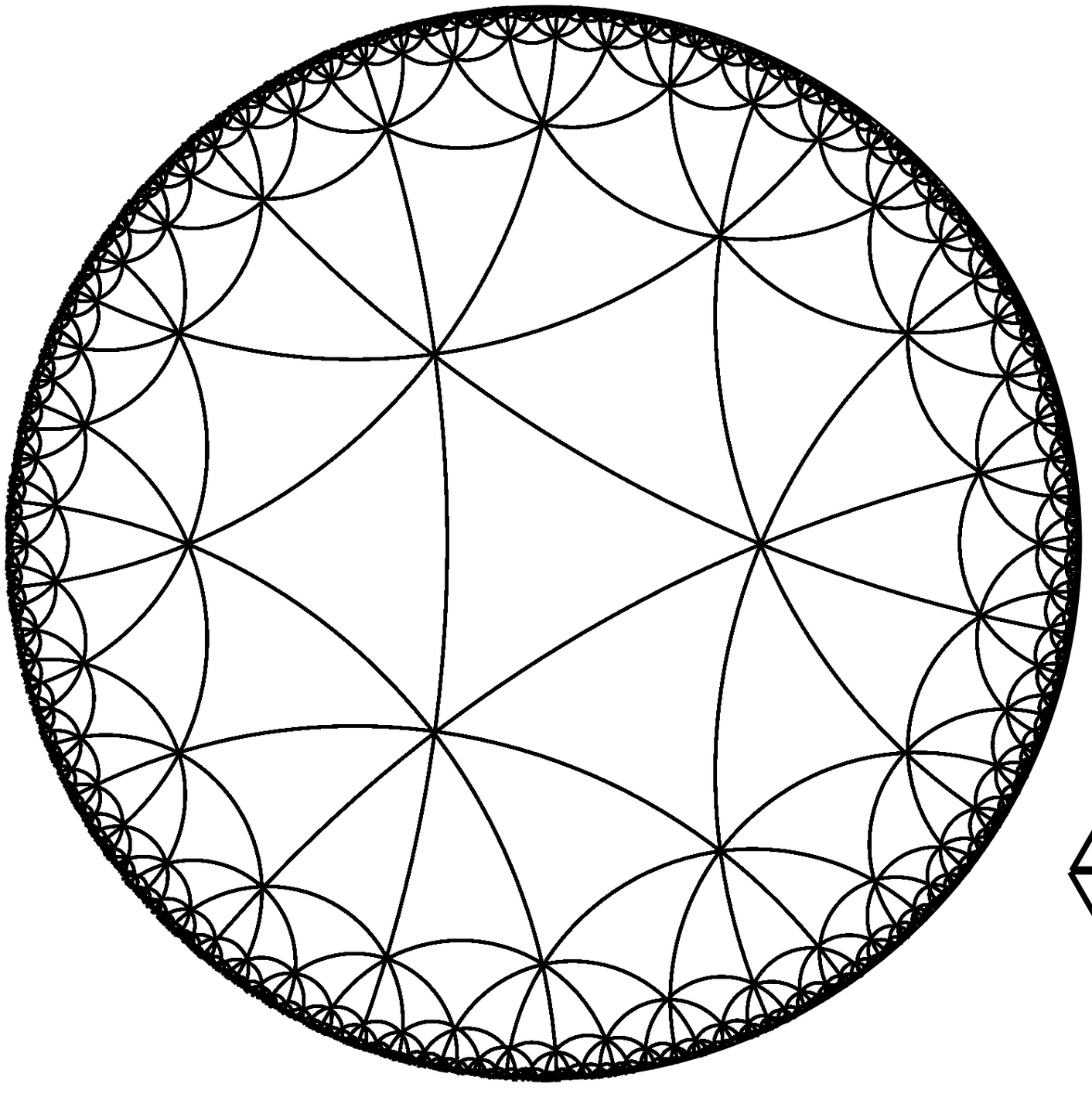
D. Infinity

Answer: $1/p + 1/q < 1/2$ implies infinity!!

# $1/p + 1/q \geqq 1/2$

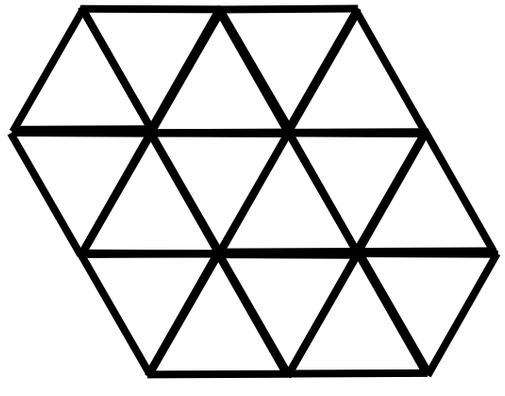
| {p,q} | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| 2 | DS | DS | DS | DS | DS | DS |
| 3 | DS | S | S | S | F | H |
| 4 | DS | S | F | H | H | H |
| 5 | DS | S | H | H | H | H |
| 6 | DS | F | H | H | H | H |
| 7 | DS | H | H | H | H | H |

# Tiling of Poincaré disk by 45-45-45 triangles

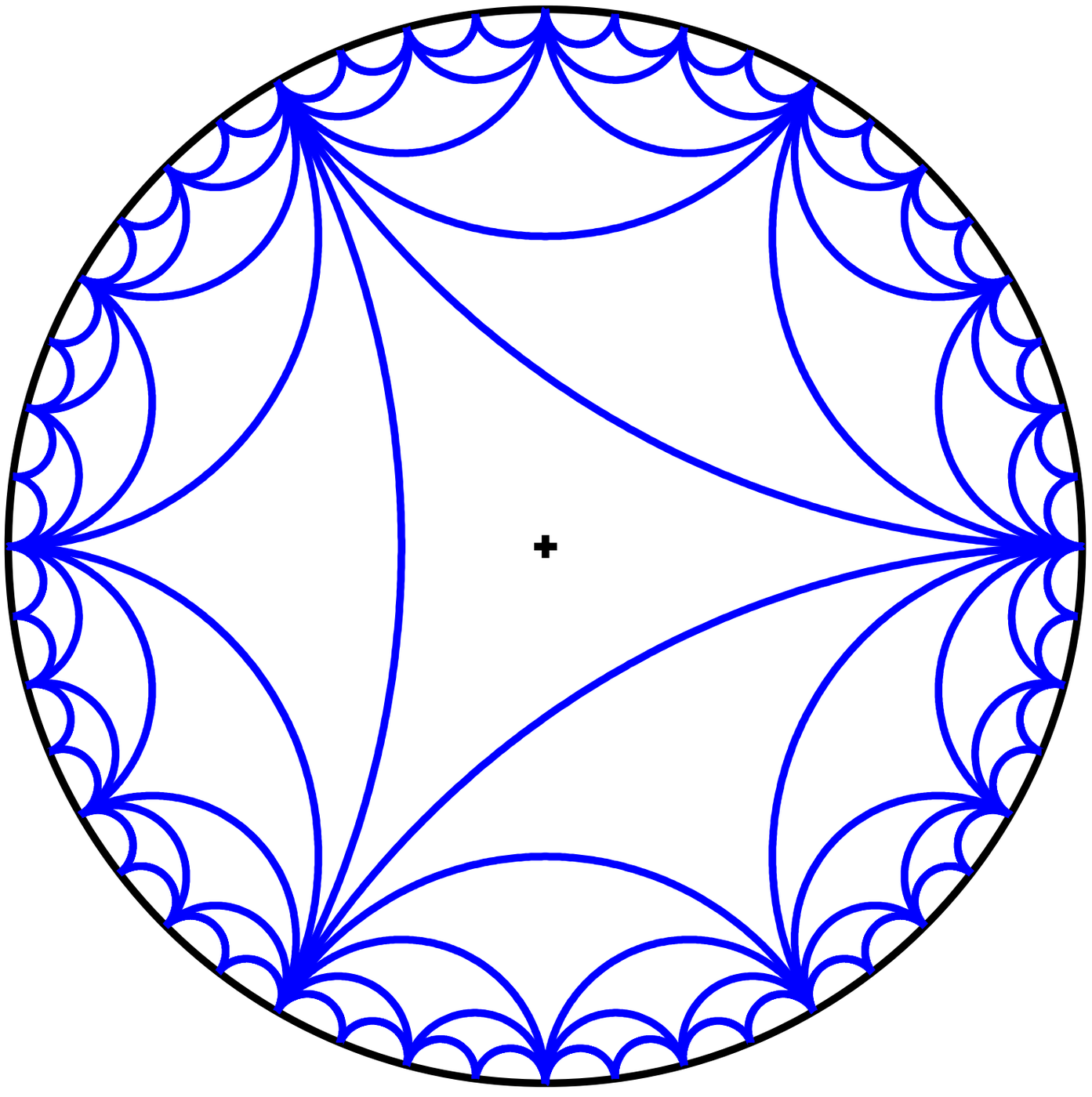


{3,8} tiling

tiling of flat space by
60-60-60 triangles



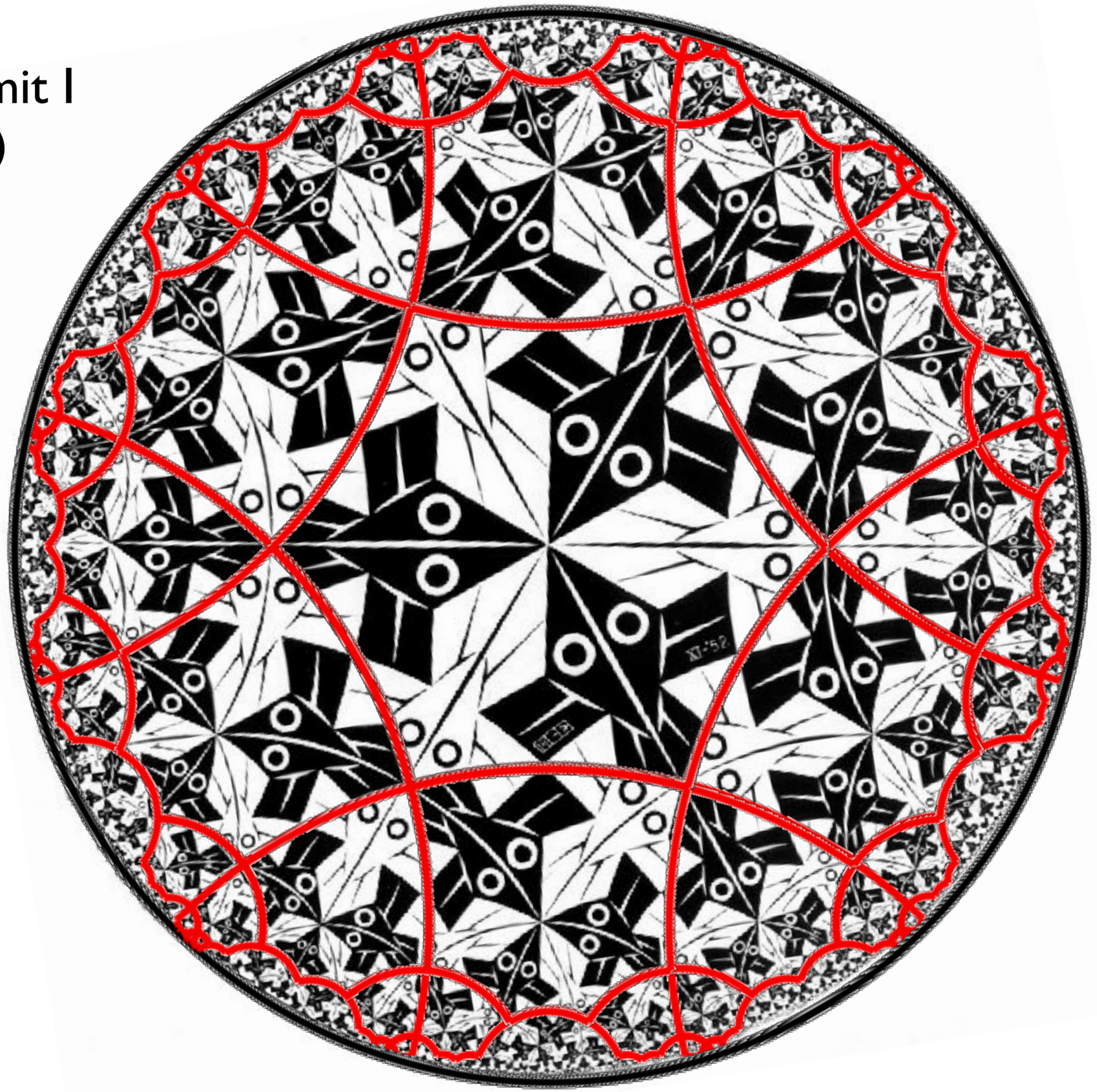
Tiling of Poincaré disk by 0-0-0 triangles!!



{3,∞} tiling

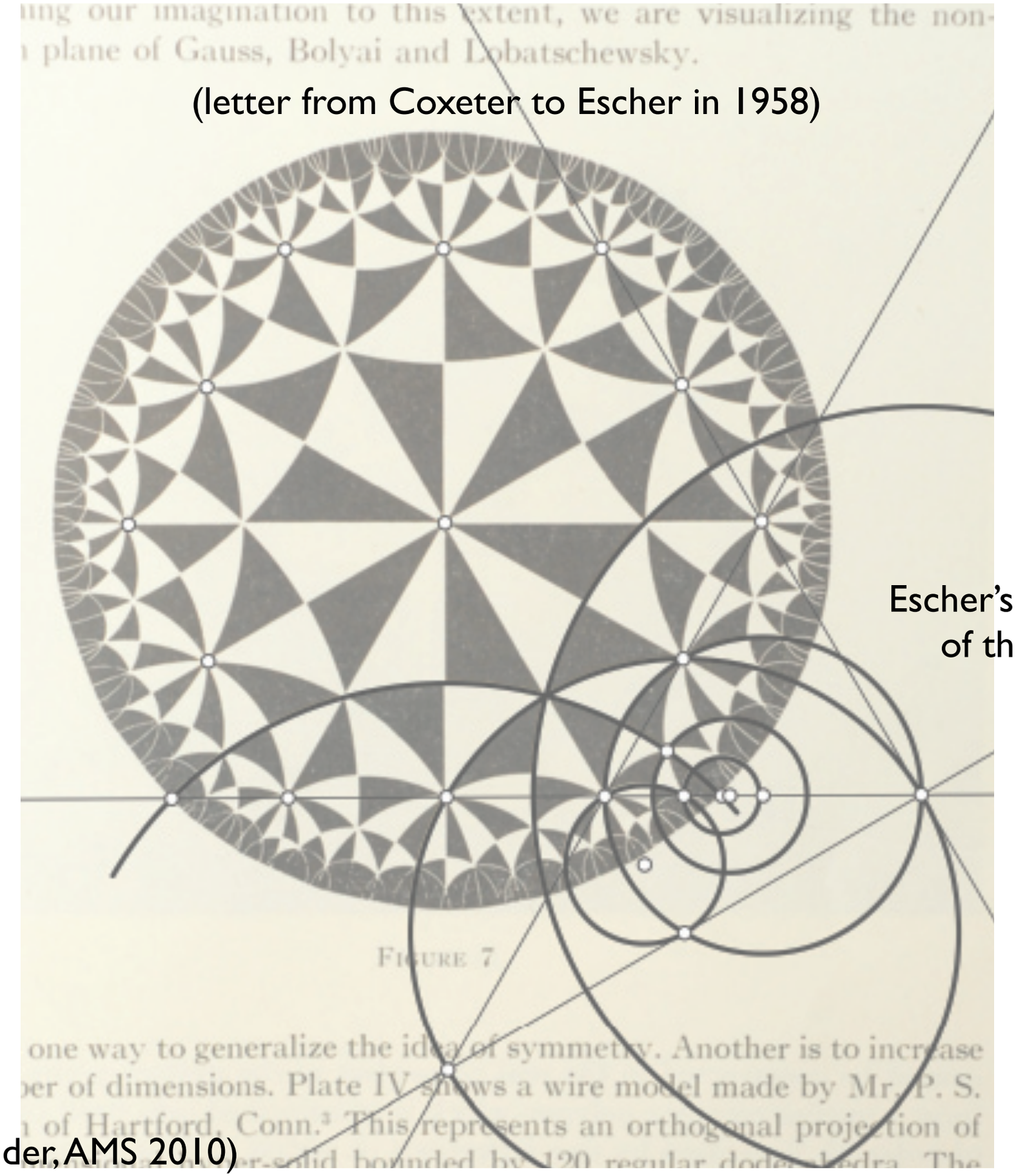
# Circle Limit I (1958)

{6,4} tiling



ing our imagination to this extent, we are visualizing the non-
plane of Gauss, Bolyai and Lobatschewsky.

(letter from Coxeter to Escher in 1958)



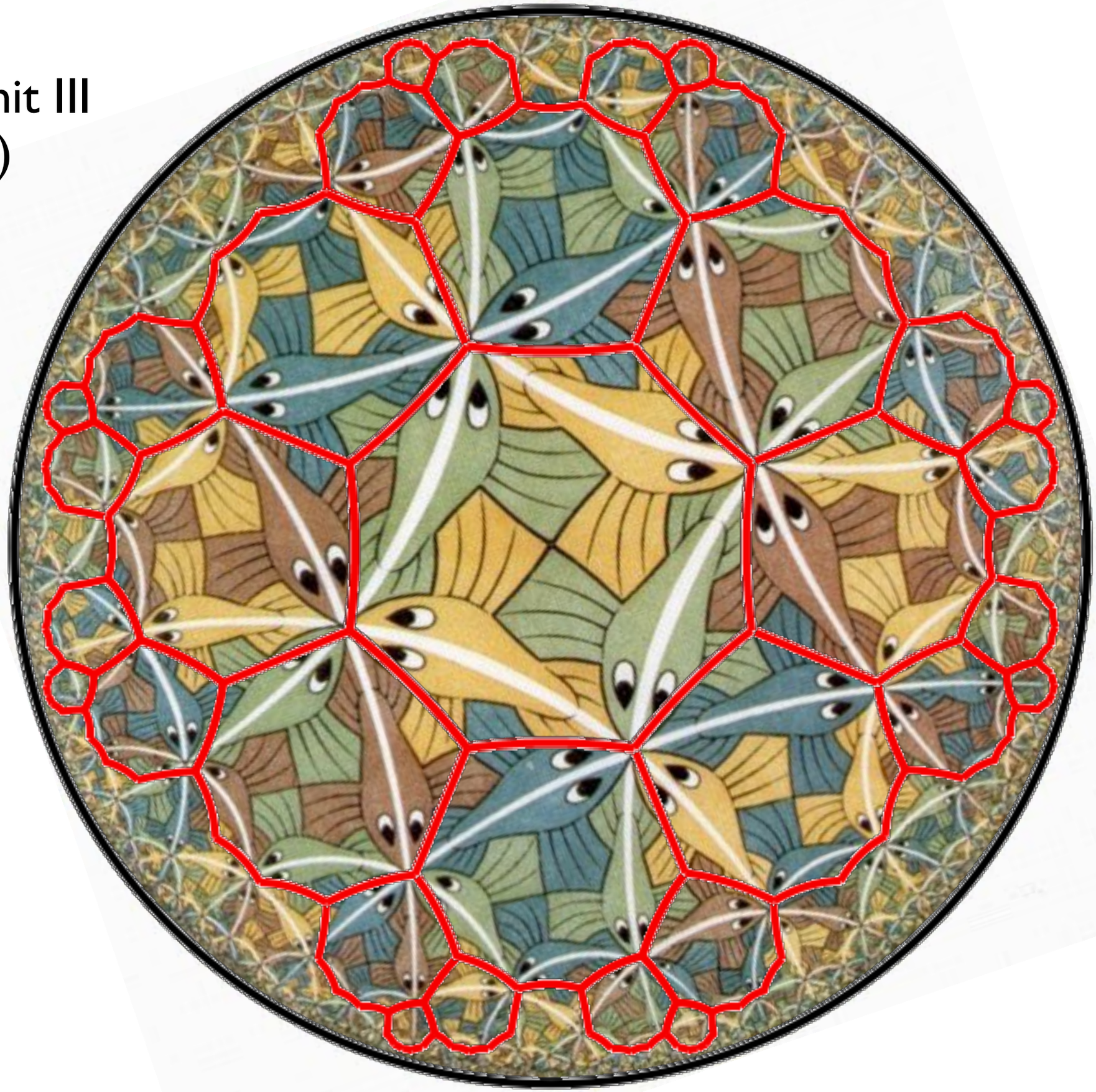
Escher's reconstruction
of the scaffolding

FIGURE 7

one way to generalize the idea of symmetry. Another is to increase
ber of dimensions. Plate IV shows a wire model made by Mr. P. S.
n of Hartford, Conn.[3] This represents an orthogonal projection of

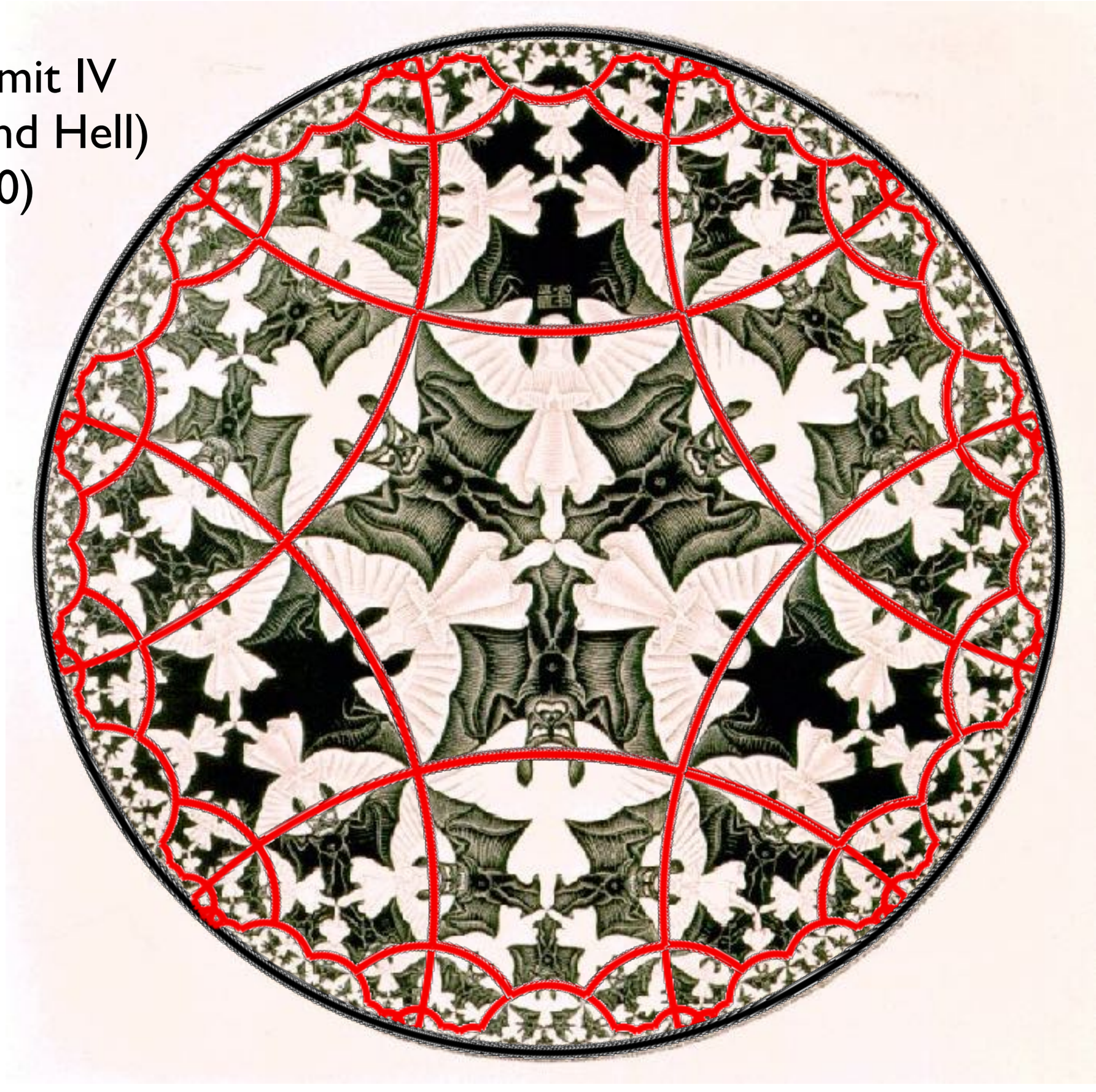
(from Schattschneider, AMS 2010)

# Circle Limit III
(1959)



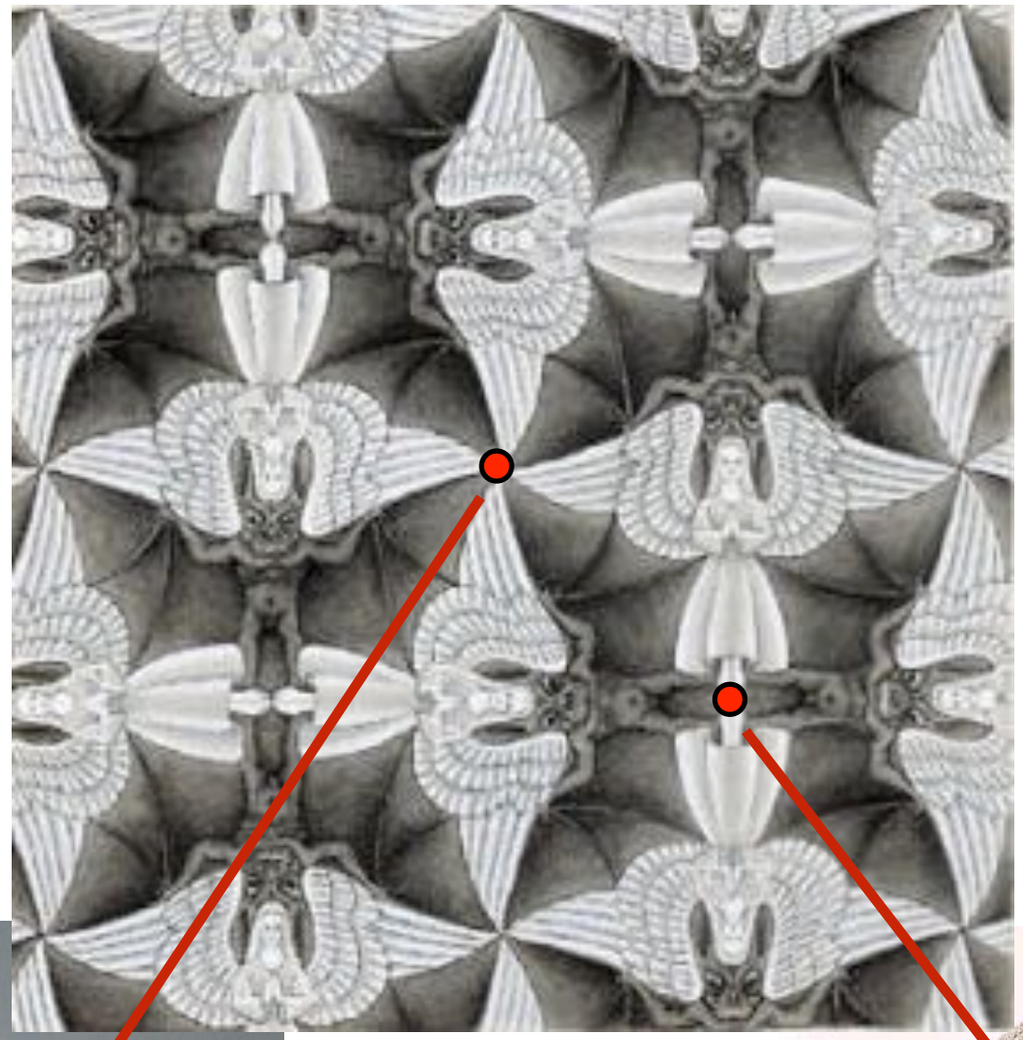
{8,3} tiling

Circle Limit IV
(Heaven and Hell)
(1960)



{6,4} tiling

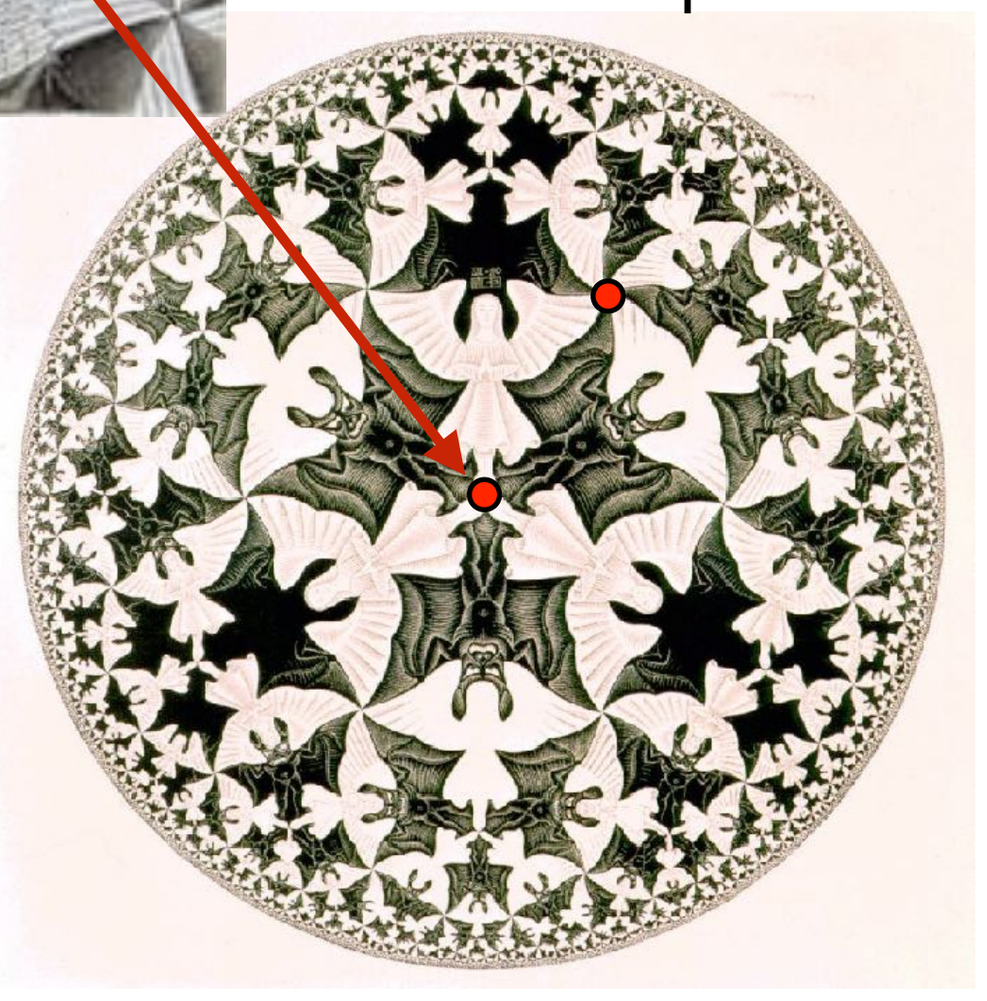
2-d flat space



2-d sphere

$1/p + 1/q \geqq 1/2$

2-d hyperbolic space



# References / credits


1. http://www.mcescher.com/ (Official M.C. Escher website)

☆ 2. http://mathcs.slu.edu/escher/ (EscherMath)

3. "M.C. Escher, His Life and Complete Graphic Work," by F.H. Bool et al.

☆ 4. "The Magic Mirror of M.C. Escher" by Bruno Ernst

☆ 5. "M.C. Escher, Visions of Symmetry," by Doris Schattschneider

6. Several mathematics articles by Doris Schattschneider and Douglas Dunham

7. ... and numerous other websites!

# extra slides

# Tafel 10. *Die 28 Grundtypen des Flächenschlusses*

| Netzecken | 6 | 5 | | | 4 | | | 3 | | |
|---|---|---|---|---|---|---|---|---|---|---|
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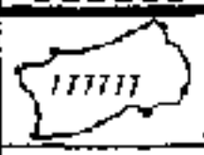
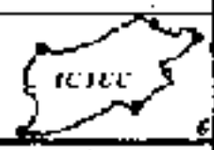
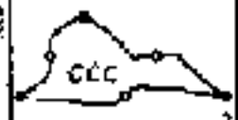
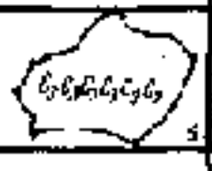
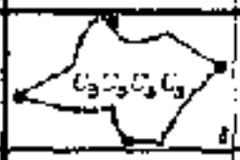
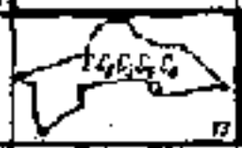
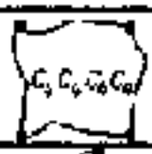
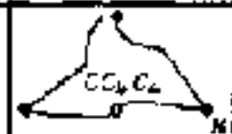
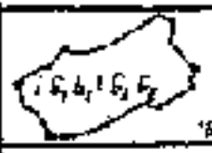
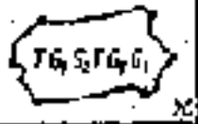
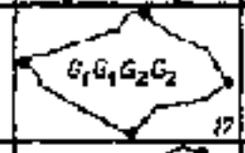
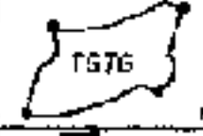
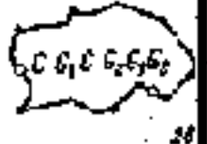
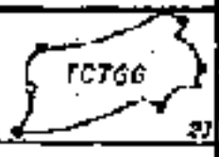
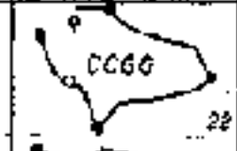
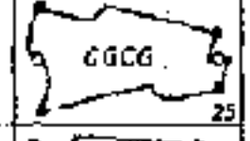
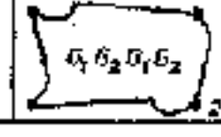
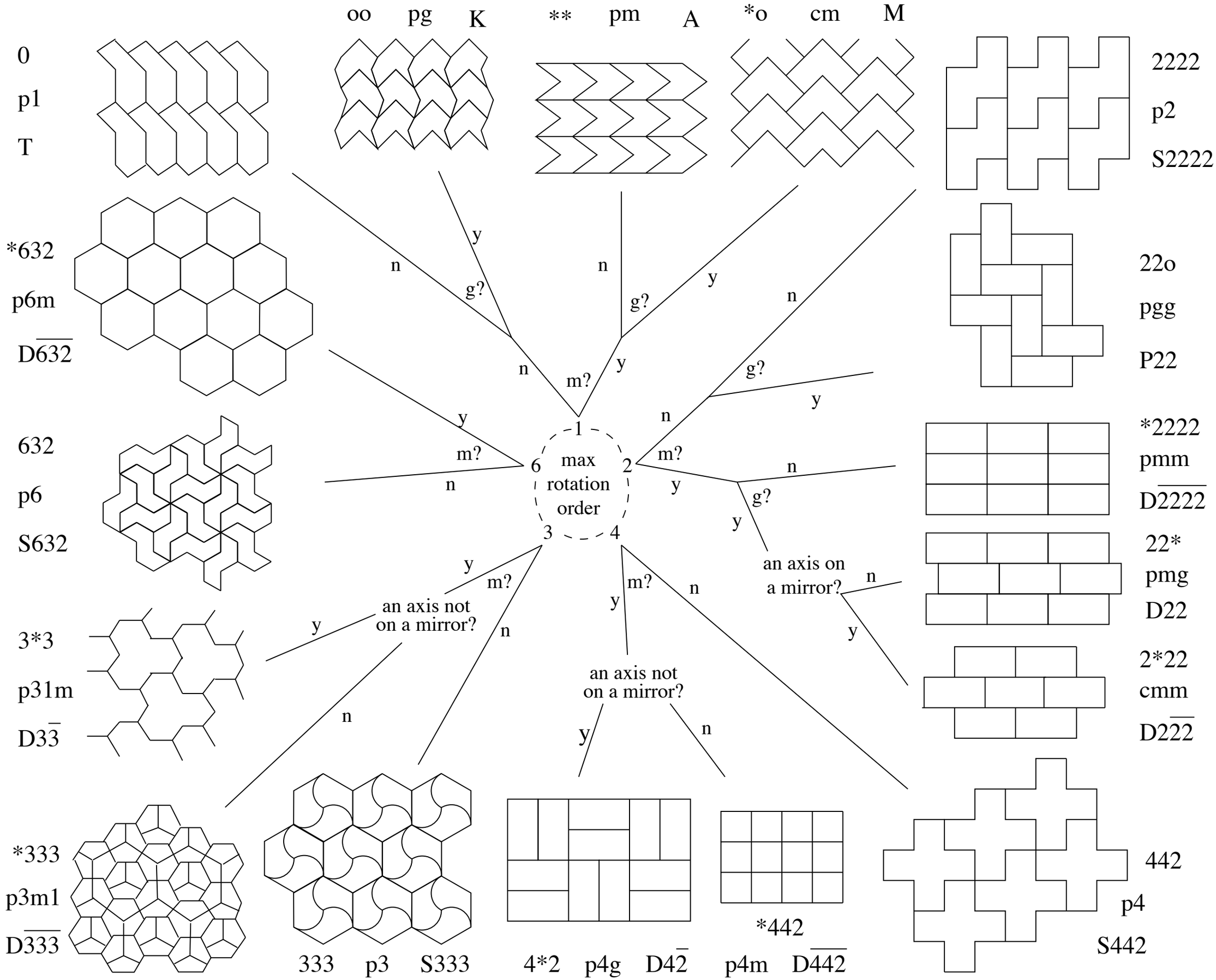
| p1 |  | | | | | | | | | |
| p2 | | | |  | | | |  | | |
| p3 |  | | | |  | | | | | |
| p6 | |  | | | | | | | | |
| p4 | | | | | | |  | |  | |
| pg |   | | | | | |   | | | |
| pgg |  | | |  | | |    | | | |

Gruppen

*Die starke Umrandung umfaßt die 9 Hauptlypen, von denen die anderen durch Schrumpfung von Linien oder Linienpaaren entstanden gedacht werden können.*

*Die Nummer rechts unten in jedem Feld ist die Nummer des zugehörigen Einzelbildes S. 64 bis 72.*

*Netzecke* —o— *Drehpunkt einer C-Linie*

# 17 wallpaper symmetry patterns

(from Brian Sanderson's webpage)