Three-body losses in a Fermi Gas at unitarity

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Introduction

Strongly correlated quantum systems are at the heart of some of the most challenging problems in contemporary physics. However, they are very difficult to treat theoretically, both from an analytical and from a numerical point of view. In the meantime, recent understanding of light-matter interaction allows experimentalists to produce ultracold atom gases with a broad degree of control of inter-species interactions, opening the door to the study of well-controlled strongly correlated systems. In some cases, the properties of such systems are universal and experimental results can be directly applied to explain the behaviour of natural materials.

Once interactions between particles from a many-body system are strong, non-trivial correlations between them emerge and lead to fascinating phenomena. In this work, we will detail a procedure that allows one to probe many-body correlations and their relations to few-body physics in the context of ultracold atoms. We will explain how the measurement of losses that occur when atoms are confined in a trap can be exploited to extract information about short-range correlations.

To begin with, in Chapter 1, we will start by reviewing phenomenological aspects of the physics of ultracold atoms and motivate the introduction of a physical quantity named the contact parameter which encapsulates information about short-range correlations of the system. Via an experiment performed on a weakly-coupled Bose-Fermi gas, we will motivate the interest in this quantity and explain how to generalize their work in order to probe three-body correlations.

Then, in Chapter 2, we will introduce the theoretical tools required to understand the two-body and three-body contact parameters by inspecting the two-body and the three-body problem.

In Chapter 3, we will reveal the link between many-body correlations and atomic losses. We will treat the specific case where atoms are confined in a harmonic trap, since we want to model future measurements to be performed in our group, using an ultracold Fermi gas of $^{40}\text{K}$ at unitarity.

Finally, in Chapter 4, we present different strategies to analyse data, namely the atom number decay and the temperature evolution, in order to determine to what accuracy we can make a quantitative prediction about three-body losses as function of the noise in the data. We will discuss their relevance according to the experimental context, in order to choose the best option to produce a quantitative prediction.
Chapter 1

Context

This first chapter will establish the motivations to study many-body correlations in a Fermi gas using atomic losses. To begin with, we shall explain phenomenological aspects of ultra cold atoms that are exploited in this work. We will see how atomic losses were already used to probe many-body correlations, and then how this work can be generalized to motivate the work presented here. Finally, a short introduction of the Fermix experiment will be presented.

1.1 Using cold atoms to produce strongly correlated systems

Ultracold atoms allow one to probe the effects of quantum statistics. Indeed, at ultracold temperatures, the description of a vapor of ultracold atoms as distinguishable classical particles is no longer valid because their wave nature can not be overlooked any more. When the de Broglie wavelength of atoms is comparable to the average distance between particles, atoms have to be treated as an ensemble of indistinguishable bodies whose collective properties are ruled by quantum statistics. When particles interact strongly, these collective properties lead to non-intuitive macroscopic phenomena. We shall explain how interactions are controlled in cold atoms experiments, and in particular how to achieve strongly-correlated regimes. We will focus on a particular regime so called unitary regime.

1.1.1 Feshbach resonance

As we shall see later in more details, the low temperature properties of interactions between atoms can be taken into account by a single parameter $a$, so called scattering length. In other terms, the fine details of the interacting potential are unimportant, and two potentials associated with the same scattering length will produce the same low energy physics: this is universality.

The scattering length enables us to control whether interactions are attractive or repulsive, and also whether they are strong or weak. Therefore, a crucial point in cold atoms is the possibility to tune the scattering length using a Feshbach resonance. In order to be concise, we will not develop the origin of Feshbach resonances here, however we will simply take their existence for granted\(^1\). In fact, it turns out that the scattering length is sensitive to the magnetic field close to a resonance at a value $B_0$ of the magnetic field. The scattering length evolves as

$$a(B) = a_{bg} \left(1 - \frac{\Delta B}{B - B_0}\right)$$

\(^1\)Extended details can be found in \[\text{[1, 2]}\]
where \( a_{bg} \) is the value of the scattering length far from a resonance, and \( \Delta B \) characterises the width of the resonance. As can be seen from the Figure 1.1 by varying the magnetic field, we can set the value of \( a \) to be positive, negative, zero, or even \( |a| \to \infty \). On each side of the resonance, the underlying physics is rich. In this problem, we will be particularly interested in the unitary regime, where \( |a| \to \infty \) i.e. when \( B \) is tuned as close as possible to a resonance \( B_0 \).

1.1.2 Unitary regime

An interesting regime for ultracold atoms is when the value of \( k_F|a| \) is of the order or larger than 1, i.e. when the scattering length is of the order or larger than the interparticle distance, which is also larger than the range of the interatomic potential. In such situation, the system is both dilute and strongly interacting, which is unusual. The limit \( k_F|a| \to \infty \) is called the unitary regime and is particularly interesting. Indeed, as the scattering length scale drops out of the problem, the only relevant length scales remaining are the inverse Fermi wave vector \( k_F^{-1} \) and the thermal wavelength \( \lambda_{th} \) (assuming that the range of the potential is very small). As a consequence, despite being strongly interacting, the system is described by universal properties and some of them can be inferred by scaling arguments.

1.2 Using losses to probe many-body correlations

In this section, we will go through an example that shows how many-body correlations can be probed by measuring atomic losses. This example is very important since this work is a generalization of this procedure.

1.2.1 Three-body collisions

At extra-low temperatures, the most thermodynamically stable phase of the system is a solid. This implies that two atoms might form a dimer after a scattering process in order to minimize the energy of the system. Two-body collisions between atoms can not result in the production of a deeply bound dimer since this can not realise simultaneously the conservation of energy and the conversation of momentum. A three-body process is possible though, as the large binding energy released by the dimer production can be transferred into kinetic energy to the newly produced dimer and the remaining partner. Since the internal energy of the produced molecule is very large compared to the temperature of the
system, the acquired kinetic energy allows them to escape from the trap, resulting in a diminution of the number of trapped atoms over time, see Figure 1.2.

![Figure 1.2: a) Plot of a typical interacting potential between atoms. The horizontal lines represent the energies of the bound states. The energies of the deeply bound states are order of magnitudes bigger than the temperature of ultracold systems. b) Representation of a cold atoms gas in a trapping potential. Three-body collisions can occur and lead to formation of a dimer (in blue). After a collision, the dimer and the remaining atom are lost.](image)

According to the process by which collisions occur, the temperature of the system might also evolve over time. If moreover losses are too fast a quasi-thermal equilibrium cannot be maintained in the cloud as well.

### 1.2.2 Example: the weakly-coupled Bose-Fermi gas

We will present an important result from the study of inelastic losses in a mixture of spinless bosons and resonant spin $1/2$ fermions with tunable inter-species interaction [3]. They produced a gas where a small number of bosons, treated as impurities, interact weakly with a gas of strongly interacting fermions. At low energy, fermions with identical spin can not interact in the s-channel because of Pauli exclusion principle (see Section 2.1). Therefore, we expect the main three-body collisions to occur between one spin-up fermion, one spin-down fermion and one boson. Now, the argument to relate many-body correlations to losses consists in saying that the recombination rate is proportional to the probability of having the three particles within a distance $b$ from each other, where $b$ is the typical size of the deeply-bound dimer. If the bosons and the fermions are tuned to be weakly interacting, then this recombination rate is proportional to the probability of having two opposite spin fermions close to each other, characterised by the so-called Tan’s Contact parameter $C_2$, named after its introduction by Shina Tan [4, 5]. In fact, if we denote $n_b$ the bosonic density, we have

$$\frac{dn_b}{dt} = -\gamma C_2 n_b$$  \hfill (1.2) \]

where $\gamma$ is a numerical factor that depends on the coupling to the non-resonant states and does not depend on the temperature or external magnetic field. The contact $C_2$ contains information about the many-body correlations of the fermions. According to the value of the scattering length, we expect different regimes for the dependence of the loss rate with respect to the density of fermions $n_f$. They are represented in the Figure 1.3. When $a < 0$, fermions of opposite spins are loosely bound into Cooper pair whose size is larger than interparticle distance. A recombination happens between one boson and two fermions, the bosonic loss rate scales as $\dot{n}_b/n_b \sim n_f^2$. When $a > 0$, fermions of opposite spins are strongly paired to form molecules. Losses occur via two-body collision between one boson and a molecule and we have $\dot{n}_b/n_b \sim n_m/2$, where $n_m = n_f/2$ is the molecule density. Finally, in the strongly correlated unitary regime $|a| \to \infty$, the loss rate is predicted to scale as
This fact tells us that in this regime, we cannot think of recombinations as isolated events occurring between a well determined amount of bodies. We cannot say, unlike the previous cases, that the boson scatters with $4/3$ fermions. This is a striking example of strong correlations inside the system, revealed in atomic losses.

All these scaling have been confirmed experimentally using a mixture of $^7\text{Li}$ and $^6\text{Li}$, and can be derived theoretically within a unified procedure by computing Tan’s contact $C_2$ [3]. This example shows how two-body correlations can be probed in a many-body system. In this work, we want to generalize this work to probe three-body correlations in a many-body system.

1.3 Generalization: the three-body contact $C_3$

A straightforward generalization of the previous study is the following: what if, instead of considering scattering between two fermions and a boson, we consider scattering between three fermions only? Therefore, the recombination rate should be proportional to the probability of having all three fermions within a distance $b$ from each other, $b$ still being the typical size of the deeply-bound molecule. This motivates the definition of a three-body contact $C_3$, which encloses the short-range behaviour of the three-body correlations in the many-body ensemble, and will be shown later to govern the loss rates in such an experiment

$$\frac{dn}{dt} = -\gamma C_3$$  \hspace{1cm} (1.3)

As we will see in this work, a very rich physics emerges from the three-body problem, which predicts exotic dependence of the atom number decay in terms of density, temperature, and scattering length. For instance at low temperature and at unitarity, we expect to have

$$\frac{dn}{dt} \propto -n^{2.85...}$$  \hspace{1cm} (1.4)

1.4 Fermix experiment

In order to measure atomic loss rates, we will use the Fermix machine, shown in Figure 1.4. The goal of the fermix experiment (Fermionic Mixture) is to produce mixtures of ultracold fermions containing two kind of species: $^6\text{Li}$ and $^{40}\text{K}$. These species are cooled
down to quantum degeneracy, i.e. a regime where classical Boltzmann statistics no longer apply. In the near future, measurement of the $^{40}\text{K}$ three-body loss rate will be performed at unitarity, that is to say when the magnetic field is tuned right on a Feshbach resonance at $B_0 = 202.10$ Gausses. The $^6\text{Li}$ will not be used for this project. Since the measurements are not performed yet, we will not describe the procedure to measure the loss rate in details.

Figure 1.4: Computer-aided design of the Fermix vacuum assembly. A Zeeman slower for $^6\text{Li}$ and a two-dimensional magneto-optical trap for the $^{40}\text{K}$ serve as atom source for the dual-species magneto-optical trap inside the octagonal chamber. A magnetic transport transfers the cloud to the glass science cell, where the quality of vacuum is better and accurate measurements can be performed.
Chapter 2

Scattering theory, from two to three-body

In this chapter we will introduce several concepts and definitions of crucial importance for the following chapters. In the first part, we will start by recalling fundamentals about two-body scattering in the context of the zero-range model which is useful to describe most cold atoms experiments. Then, we will see how this construction generalizes in the case of a three-body system. In the second part, we will introduce the two-body and the three-body contact parameters and explain how they are related to the short-range correlations of the system.

2.1 Two-body scattering in the zero-range model

2.1.1 The two-body problem

Let us start by treating the problem of two interacting isolated particles. We consider the Hamiltonian of two particles of mass $m$ interacting via a potential $V(r)$ in 3D

\[ \hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + V(\hat{r}_1 - \hat{r}_2) \]  

(2.1)

The interaction potential is such that only relative distance between particles matter $V(r) = V(|r|)$. It is convenient to study the problem in the center of mass frame with new variables

\[
\begin{align*}
R &= \frac{1}{2}(r_1 + r_2) \\
r &= r_1 - r_2
\end{align*}
\]

(2.2)

The Hamiltonian now reads

\[ \hat{H} = \frac{\hat{P}_1^2}{2M} + \frac{\hat{P}_2^2}{2m_r} + V(r) \]  

(2.3)

The first term describes the total energy of a free particle of total mass $M = 2m$ whereas the two other terms describe the total energy of a particle of reduced mass $m_r = m/2$ moving in a central potential $V(r)$. The physics of two-body scattering is described exclusively by these two last terms. The corresponding Schrödinger equation is

\[ \left[ -\frac{\hbar^2}{m} \nabla_r^2 + V(r) \right] \psi(r) = E\psi(r) \]  

(2.4)

Using spherical coordinates, one can separate variables

\[ \psi(r) = R_l(r)Y_{l,m}(\theta, \phi) \]  

(2.5)
Here, $Y_{l,m}(\theta, \varphi)$ denotes the spherical harmonics and the radial part of the wavefunction. Therefore, $R_l(r)$ obeys

$$-\frac{\hbar^2}{m} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR_l}{dr} \right) + V(r)R_l(r) + \frac{\hbar^2}{m} l(l+1)R_l(r) = ER_l(r) \quad (2.6)$$

This form of the equation shows a centrifugal barrier term proportional to $l(l+1)$ where $l$ is the quantum number associated to the orbital angular momentum. In most situations relevant in ultracold atoms experiments, the centrifugal barrier term for $l \neq 0$ is big compared to the kinetic energy of the atoms. Therefore, in the low-energy limit, only scattering events in the $l = 0$, or $s$-wave channel are important.

The zero-range model consists in considering that the potential $V(r)$ is zero whenever $r \neq 0$. The effect of the potential on the system is fully taken into account via the so called Bethe-Peierls boundary condition

$$\begin{cases} 
-\frac{\hbar^2}{m} \Delta_r \psi(r) = E \psi(r) & \text{if } r \neq 0 \\
\psi(r) \rightarrow A \left( \frac{1}{r} - \frac{1}{a} \right) & r \rightarrow 0
\end{cases} \quad (2.7)$$

Where $a$ is a constant that depends on the interacting potential and $A$ is here for normalization. It is important to realize that $a$, the scattering length, is the only parameter necessary to describe the system.

If $a > 0$, there exists a bound state, i.e. a state solution of the Schrödinger equation with a negative energy $E = -\frac{\hbar^2}{ma^2} < 0$ associated with a wavefunction whose spatial extension has a characteristic length $a$

$$\psi(r) = \frac{e^{-r/a}}{r\sqrt{2\pi a}} \quad (2.8)$$

The scattering states, i.e. with a positive energy $E > 0$ in the $s$-wave channel have the form

$$\psi(r) = e^{ikr} + f_k \frac{e^{ikr}}{r} \quad (2.9)$$

where $f_k$, the scattering amplitude, is

$$f_k = -\frac{1}{a + ik} \quad (2.10)$$

The differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = |f_k|^2 \approx a^2 \quad \Rightarrow \quad \sigma = 4\pi a^2 \quad (2.11)$$

In fact, if we deal with identical particles, we have to symmetrize or anti-symmetrize the wavefunction to take into account the fact that we deal with bosons or fermions. This implies that the scattering amplitude satisfies

$$\frac{d\sigma}{d\Omega} = |f_k(\theta, \varphi) + \eta f_k(\pi - \theta, -\varphi)|^2 \quad (2.12)$$

where $\eta = +1$ for bosons, and $\eta = -1$ for fermions. In the $s$-channel, $f_k$ is isotropic so we have $\sigma = 8\pi a^2$ for identical bosons, and $\sigma = 0$ for identical fermions. This explains why two fermions with the same internal state do not interact via the $s$-channel, due to Pauli exclusion principle.
2.1.2 Asymptotic form of the many-body wavefunction

In this section, we explain how the many-body wavefunction behaves when two particles approach each other, using the knowledge exposed before. Here, we are interested in an assembly of $N$ unpolarized spin-1/2 fermions consisting of $N_{\uparrow} = N/2$ spin-up and $N_{\downarrow} = N/2$ spin-down. The many-body wavefunction is anti-symmetric with respect to the exchange of any pair of fermions with identical spin. Once again, we can use a zero-range model to describe the system. This implies that whenever there is no contact between particles, the Schrödinger equation is

$$\left( -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta r_i \right) \Psi(r_1, \ldots, r_N) = E \Psi(r_1, \ldots, r_N)$$  \hspace{1cm} (2.13)$$

As before, we account for contact interactions by imposing Bethe-Peierls boundary conditions. If particles 1 (spin-up) and 2 (spin-down) get close to each other, we need to have

$$\Psi(r_1, \ldots, r_N) \sim \left. \frac{1}{r} \left( \frac{1}{a} - 1 \right) \right| \frac{A(r_c, r_3, \ldots, r_N)}{r \to 0}$$ \hspace{1cm} (2.14)$$

where $r = |r_1 - r_2|$ is the relative distance between the two particles, $r_c$ is the position of the center of mass of the particles 1 and 2. This limit holds when all other particles are fixed. This means that the relative motion between the two particles can be factored out from the wave function, and all the remaining dependence and many-body effects are enclosed in the function $A(r_c, r_3, \ldots, r_N)$. The contact condition for every other pair of particle is ensured by the antisymmetry of the wavefunction.

2.1.3 Two-body contact

Shina Tan has defined a physical quantity, the two-body contact $C_2$ \[4, 5\] as

$$C_2 = \lim_{k \to 0} k^4 n_\sigma(k) \quad \sigma = \uparrow, \downarrow$$ \hspace{1cm} (2.15)$$

where $n_\sigma(k)$ is the momentum distribution. Besides being related to loss rates as observed in the weakly coupled Bose-Fermi gas experiment, Shina Tan has shown that this quantity plays an important role because in other physical quantities. For instance, it is related to the total energy of system through the adiabatic sweep theorem. For a two-spin component fermions, it reads

$$\frac{dE}{d(1/a)} = -\frac{\hbar^2 C_2}{4\pi m}$$ \hspace{1cm} (2.16)$$

The two-body contact can also be defined in a more abstract way via the integral of the regular part of the asymptotic form of the many-body wavefunction when two interacting particles are close to each other.

$$C_2 = (4\pi)^2 N_{\uparrow} N_{\downarrow} \int d^3r_c d^3r_3 \ldots d^3r_N |A(r_c, r_3, \ldots, r_N)|^2$$ \hspace{1cm} (2.17)$$

Therefore, it encloses the influence of the many-body environment on the asymptotic two-body physics. In a uniform system, the two-body correlation function $g^{(2)}(r_\uparrow, r_\downarrow)$ can be expressed as

$$g^{(2)}(r_\uparrow, r_\downarrow) = \left. \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\uparrow(r_\uparrow) \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow(r_\downarrow) \rangle \right|_{|r_\uparrow - r_\downarrow| \to 0} = \frac{C_2}{4\pi |r_\uparrow - r_\downarrow|^2}$$ \hspace{1cm} (2.18)$$

Now that the relation between the two-body contact $C_2$ and two-body correlations have been measured and are well understood, it is natural to generalize this concept for three-body correlations.
2.2 Three-body scattering theory at unitarity

2.2.1 The three-body problem

Now we present results of a similar theory for a system consisting of three spin-1/2 fermions of same mass $m$ located at $r_1, r_2$ and $r_3$ \[7, 8\]. Since we need the fermions to interact via the s-wave channel, we suppose that we have two spin-up fermions, labelled by 1 and 3, and one spin-down fermion labelled by 2. We can characterize the system by the position of the center of mass $R_c$ and the relative position of particle, using Jacobi coordinates $(r, \rho)$, see Fig. 2.1

\[
R_c = \frac{r_1 + r_2 + r_3}{3}
\]

we can use a trick similar to the one we used for the two-body problem in order to decouple the center of mass motion and the relative motion. We obtain a Schrödinger equation for the relative motion. In the zero range model, it is

\[
\left[ -\hbar^2 \frac{\Delta_r + \Delta_\rho}{2m} \right] \psi(r, \rho) = E \psi(r, \rho)
\]

when the particles are not in contact, i.e $r \neq 0$ and $\pm r/2 + (\sqrt{3}/2)\rho \neq 0$. Just like before, the effect of the interactions is fully taken into account by imposing a Bethe-Peierls condition for each pair of particles. In fact we only need to impose

\[
\psi(r, \rho) = \frac{1}{r} A(\rho) + O(r)
\]

and the conditions on the other pairs are automatically set by the symmetry of the wave function $\psi(r_1, r_2, r_3) = -\psi(r_3, r_2, r_1)$.  

Efimov ansatz It is possible to solve the problem in the unitary regime where $1/a \to 0$. We can define hyperspherical coordinates $(R, \Omega)$, where $R$ is the hyperradius and $\Omega = (\alpha, \hat{r}, \hat{\rho})$ a set of hyper-angles

\[
R = \sqrt{\frac{r^2 + \rho^2}{2}} \quad \text{and} \quad \begin{cases} \alpha = \arctan(r/\rho) \\ \hat{r} = r/r \\ \hat{\rho} = \rho/\rho \end{cases}
\]

therefore, the Schrödinger equation becomes

\[
-\frac{\hbar^2}{2m} \left[ \frac{1}{R^2} \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} + \frac{1}{R^2} T_\Omega \right] (R^2 \psi(R, \Omega)) = E \psi(R, \Omega)
\]
where $T_\Omega$ in an operator which encodes all the angular dependence of the hamiltonian. Efimov has shown that the radial and the angular dependence can be separated [7] and hence the wave function can be written

$$\psi(R, \Omega) = \frac{F(R)}{R^2} \varphi(\Omega)$$  \hspace{1cm} (2.24)$$

It turns out that in this regime, the problem can be solved analytically. To begin with, consider the hyper-angular problem

$$T_\Omega \varphi_{\ell,m,n}(\Omega) = -s_{\ell,n}^2 \varphi_{\ell,m,n}(\Omega)$$  \hspace{1cm} (2.25)$$

The eigenvalues $s_{\ell,n}$ are labelled by two quantum numbers $\ell, n \in \mathbb{N}$ where $\ell$ represents the total relative angular momentum and $n$ just labels the allowed solutions. The integer $m \in [-\ell, \ell]$ labels the multiple degenerate eigenfunctions associated with a given $\ell$ and $n$, exactly like the two-body problem. They are solution of the transcendental equation

$$\left[ \sum_{k=0}^{\ell} \frac{(-\ell)_k (\ell+1)_k (1-s)_\ell}{k! (1-s)_k} \left( 2^{-k+i(k-s)}e^{i\pi/2} - (-1)^\ell \frac{4}{\sqrt{3}} e^{-i\pi/2} (2k+s) \right) \right] - \left[ \sum_{k=0}^{\ell} \frac{(-i)_k (\ell+1)_k (1-s)_\ell}{k! (1-s)_k} \left( 2^{-k-i(k-s)}e^{-i\pi/2} - (-1)^\ell \frac{4}{\sqrt{3}} e^{i\pi/2} (2k+s) \right) \right] = 0$$  \hspace{1cm} (2.26)$$

where the Pochhammer symbol is defined by

$$(x)_n = x(x+1)\ldots(x+n) \quad \text{and} \quad (x)_0 = 1$$  \hspace{1cm} (2.27)$$

Several numerical values of $s_{\ell,n}$ are given in Table 2.1. Once the hyper-angular problem is solved, one can write the equation for the hyper-radial part

$$-F''(R) - \frac{1}{R} F'(R) + \frac{s_{\ell,n}^2}{R^2} F(R) = \frac{2mE}{\hbar^2} F(R)$$  \hspace{1cm} (2.28)$$

Once again, we identify a centrifugal term governed by the value of $s_{\ell,n}$. For fermionic systems, all the $s_{\ell,n}$ are real positive number so there is a centrifugal barrier. This implies that there exists only scattering states $F(R) = AJ_{s_{\ell,n}}(kR)$, with $k^2 = 2mE/\hbar^2$ where $J_{s_{\ell,n}}$ is a Bessel function. In the following we will be interested in short-range behaviour of the radial wavefunction, governed by the value of $s_{\ell,n}$

$$F(R) \sim (kR)^{s_{\ell,n}}$$  \hspace{1cm} (2.29)$$

We should mention that for bosonic systems, a similar procedure can be used to solve the three-body problem and it turns out that the value of $s_{\ell,n}$ for $\ell = n = 0$ is a pure imaginary number $s_{0,0} \approx 1.0062i$. Therefore, the centrifugal term in Eq (2.28) is an attractive potential that implies the existence of a three-particles bound state. This very fact modifies completely the treatment of the problem and leads to a rich Efimov physics which will not be covered at all here.
2.2.2 Description in the many-body problem

Let us come back to our many-body problem defined by Eqs. (2.13) and (2.14). We can directly apply the knowledge of the three-body problem to determine the asymptotic form of the many-body wavefunction when three-particles get very close to each other, all other particles being fixed. It takes the following form

\[ \Psi(r_1, ..., r_N) = \frac{1}{R^{2s_{\ell,n}}} \sum_{m=-\ell}^{\ell} \phi_{\ell,n,m}(\Omega) B_m(R_c, r_4, ..., r_N) \quad (2.30) \]

Again, the wavefunction is factorized into a first factor that only depends on the hyper-radius \( R \) of the triplet, and a second one that contains all the other variables: the hyper-angle \( \Omega \) and center of mass position \( R_c \) of the triplet, and the position of all the other particles of the many-body system \( r_i, i > 3 \). We see that according to the channel under consideration, i.e. the value of \( s_{\ell,n} \) considered here, the divergence of the wavefunction will be different. The largest contribution comes from the smallest \( s_{\ell,n} \) which corresponds to \( \ell = 1 \) and \( n = 0 \) and whose value is \( s_{1,0} = s \approx 1.7727 \). In the following, we will consider exclusively this channel.

2.2.3 Three-body contact

By analogy with the definition of the two-body contact in Eq. (2.15), we define the three-body contact \( C_3 \) via the integral of the regular part of the asymptotic form of the many-body wavefunction when three-particles get close to each other

\[ C_3 = N_\uparrow(N_\uparrow - 1)N_\downarrow \sum_{m=-1}^{+1} \int d^3R_c d^3r_4 d^3r_5 |B_m(R_c, r_4, ..., r_N)|^2 \quad (2.31) \]

Since these two quantities are extensive, we can define their intensive analogue, so called contact density as \( C_2 = C_2/V \) and \( C_3 = C_3/V \). Measurements of the three-body contact have been performed in universal bosonic systems [9]. However, for fermionic systems, no theoretical and experimental works have been performed yet \( C_3 \). Whatever it be, \( C_3 \) is related to the losses due to three-body recombinations in a single species atomic gas.

**Relation to pair and triplet distribution** The three-body correlation function \( g^{(3)} \) gives the triplet distribution. For fermionic triplets of the form \( (\uparrow \uparrow \downarrow) \), this function is defined as

\[ g^{(3)}_{\uparrow\uparrow\downarrow}(r_1, r_2, r_3) = \langle \hat{\psi}_\uparrow^\dagger(r_1) \hat{\psi}_\uparrow^\dagger(r_2) \hat{\psi}_\downarrow^\dagger(r_3) \hat{\psi}_\uparrow(r_2) \hat{\psi}_\uparrow(r_1) \rangle \quad (2.32) \]

The number of triplets with hyper-radius smaller than \( d \) is

\[ N_{\uparrow\uparrow\downarrow}(d) = \frac{1}{2!} \int_{R<d} d^3r_1 d^3r_2 d^3r_3 g^{(3)}_{\uparrow\uparrow\downarrow}(r_1, r_2, r_3) \quad (2.33) \]

Switching to first quantization, we can use Eq. (2.30) which holds provided \( d \) is small enough, and identify the three-body contact to get

\[ N_{\uparrow\uparrow\downarrow} \sim C_3 \frac{d^{2s+2}}{2s+2} \quad (2.34) \]

Notice that we only treat the unpolarized case here, therefore, we could also define \( N_{\downarrow\downarrow\uparrow} \) and this would be totally equivalent.
Chapter 3

Atom number and temperature dynamics in a harmonic trap

The goal of this chapter is to relate the three-body contact $C_3$ to the atomic loss rate. Using scaling arguments, we will see how we expect the loss rate to depend on temperature and density at unitarity. Then we will build a model that describes the dynamics of the number of atoms and the temperature in a harmonic trap. This model will also take into account the evaporative cooling caused by the finite depth of the trapping potential.

3.1 Three-body losses

Let us relate the three-body losses and three-body contact $C_3$. We can use the same argument than the one used in Sec.1.2 to relate the losses in the weakly coupled Bose-Fermi gas with the two-body contact $C_2$. We can assume that the three-body inelastic scattering rate is proportional to the probability of having three-fermions in the configuration $(\uparrow\uparrow\downarrow)$ or $(\downarrow\downarrow\uparrow)$ within a distance $b$ from each other, where $b$ is the typical interaction range. If we assume that we have $N$ atoms contained in a homogeneous box, this implies that the atom loss rate is proportional to the number of triplets $(\uparrow\uparrow\downarrow)$ or $(\downarrow\downarrow\uparrow)$ of hyperradius smaller than $b$[10]

\[
\frac{dN}{dt} = -N_{\uparrow\uparrow\downarrow}(b) \frac{\hbar}{mb^2} K
\]

where $K$ is a dimensionless constant that depends on the short-range properties of the material but not on other parameters. Dividing by the volume and using Eq. (2.34), we already see that the loss rate is proportional to the three-body contact density $C_3$.

\[
\frac{dn}{dt} = -C_3 \frac{\hbar 2s}{m 2s + 2} K \tag{3.2}
\]

3.1.1 Scaling relations

We can use scaling arguments to relate the loss rate to density, temperature, and scattering length. We can write $C_3$ in terms of dimensionless quantities $k_Fb$, $1/k_Fa$, $1/k_F\lambda_{th}$. Here, $k_F$ is the Fermi wave vector and $\lambda_{th}$ is the thermal wavelength. From Eq. (2.34), we see that the dimension of $C_3$ is $[C_3] = [C_3]/[V] = 1/L^{2s+5}$. Therefore, we can write

\[
C_3 = k_F^{2s+5} \Lambda(1/k_F a, 1/k_F \lambda_{th}) \tag{3.3}
\]

At unitarity, we have $1/k_F a \to 0$ and at zero temperature, we have $1/k_F \lambda_{th} \to 0$ so that the $\Lambda(0, 0)$ is just a numerical coefficient provided the limits are regular. We also recall that $k_F \sim n^{1/3}$. We get

\[
\frac{dn}{dt} \sim k_F^{2s+5} \sim n^{(2s+5)/3} \approx n^{2.85...} \tag{3.4}
\]
Thus, this justifies Eq. (1.4).

However, in this work, we are interested in the temperature dependence of the loss rate in the non-degenerate regime. Combining this argument with Eqs. (3.3) and (3.2), we have

\[
\frac{dn}{dt} = k_F^{2s+5} \Lambda(0, 1/k_F \lambda_{th}) \frac{\hbar}{m} \frac{b^{2s}}{2s+2} K
\]

(3.5)

At high-temperature, the thermal wave-length \( \lambda_{th} \) is small and we can treat atoms as point-like objects. Therefore, we expect the loss rate to scale as \( \dot{n} \sim n^3 \). The density dependence is adjusted via the function \( \Lambda(0, 1/k_F \lambda_{th}) \)

\[
\frac{dn}{dt} \sim k_F^{2s+5} \left( \frac{1}{k_F \lambda_{th}} \right) \sim n^3 \sim k_F^9
\]

(3.6)

which implies \( \gamma = 2s - 4 \). Since \( \lambda_{th} \sim T^{-1/2} \), we have

\[
\frac{dn}{dt} = -\lambda_3 \frac{n^3}{T^\delta} \quad \delta = -\frac{\gamma}{2} = 2 - s \approx 0.23\ldots
\]

(3.7)

where we introduced the constant \( \lambda_3 \). We see here that the temperature dependence of the loss rate is very weak, and in the following, we will develop models and strategies to extract the values of \( \delta \) and \( \lambda_3 \) from experimental data. At this stage, it is useful to notice that the same scaling argument applied to the weakly coupled Bose-Fermi gas from \([3]\) successfully predicts the good dependence of the loss rate in each regime.

### 3.2 Three-body losses at unitarity in a harmonic trap

In our cold atoms experiment, atoms are confined in a trap that must be taken into account in the rate equations. The trap is assumed to be harmonic, inducing a potential \( U(r) \) of the form

\[
U(r) = \frac{1}{2}m(\omega_x x^2 + \omega_y y^2 + \omega_z z^2)
\]

(3.8)

and we denote \( \omega = (\omega_x \omega_y \omega_z)^{1/3} \) the geometric mean frequency. We place ourselves in the non-degenerate regime, which implies that the system is correctly described by a Boltzmann distribution. The distribution function has the following form

\[
f(p, r) = \frac{1}{Z} e^{-\beta U(r)} e^{-\beta p^2/2m} \quad Z = \frac{(2\pi k_B T)^3}{\omega^{3/2}}
\]

(3.9)

We will derive the differential equations that govern the evolution of the atom number and temperature. We will follow the procedure used to treat a unitary Bose gas from \([13]\).

**Atom number decay** As we have seen before, at unitarity in the non-degenerate regime, the atom number decay is expressed as

\[
\frac{dn}{dt} = -\lambda_3 \frac{n^3}{T^\delta} \quad \delta = 0.23\ldots
\]

(3.10)

where \( \lambda_3 \) is a temperature independent constant. We can integrate this expression over the system volume, and we obtain

\[
\frac{dN}{dt} = -\gamma_3 \frac{N^3}{T^{3+\delta}} \quad \gamma_3 = \lambda_3 \left( \frac{m\omega^2}{2\sqrt{3\pi k_B}} \right)^3
\]

(3.11)
Inspecting Eq. (3.10), we see that the losses will happen more frequently at the center of the trap, where the density is maximum. On average, the potential energy of the lost atoms is below the average potential energy $3k_B T/2$ of the total ensemble. This implies that low energy atoms escape from the trap more frequently than high energy atoms. Therefore, once the system has rethermalized, i.e. both potential and kinetic energy have been redistributed, the average total energy per particle has increased. Put in another way, this process increases the temperature of the system. If the elastic collision rate is too small, the system cannot thermalize and the phase-space density is no longer given by Boltzmann distribution Eq. (3.9). In the following, we will always assume that this characteristic time is very small compared to other time scales, so that we neglect the influence of three-body losses on the equilibrium properties of the system.

Now, let us see how temperature evolves over time. The average energy of an atom that leaves the trap due to three-body recombination is $(2 - \frac{4}{3}) k_B T$, whereas the average energy of atoms in a harmonic potential is $3k_B T$ (equipartition theorem). This result is shown in Appendix and generalizes a previous result obtained for unitary bosons [13]. Therefore, we can write a differential equation for the total energy of the system. The change in total energy during an infinitesimal time $d t$ is equal to number of atoms lost during $d t$ multiplied by the average energy carried by these atoms. Thus, we get

$$\frac{dE}{dt} = \left(2 - \frac{\delta}{3}\right) k_B T \frac{dN}{dt}$$  \hspace{1cm} (3.12)

Since we know the expression of the total energy $E = 3Nk_B T$, we can isolate the variation of temperature, and obtain

$$\frac{dT}{dt} = \frac{1}{3} \left(1 + \frac{\delta}{3}\right) T \gamma_3 \frac{N^2}{T^{3+\delta}}$$  \hspace{1cm} (3.13)

The equations (3.11) and (3.13) describe the coupled evolution of atom number and temperature due to three-body losses. However, three-body recombinations are not the only sources of losses. Depending on the experimental context, there are at least two other relevant sources:

- The vacuum inside which cold atoms are contained can not be perfect. Hot atoms can be radiated from the wall of the vacuum chamber. If one of them happens to collide with a cold atom from the trap, this atom will acquire too much kinetic energy and will be lost. These one-body losses can be modelled by adding a term $-\frac{1}{\tau} N$ in the right-hand side of Eq. (3.11), where $\tau$ is a characteristic time to be measured experimentally by looking at the decay of atom number when interactions are switched off ($a = 0$) and when the trapping potential is extremely deep.

- Evaporative cooling can happen if the depth of the trap is not too large. Fast atoms can escape the trap after a collision if their kinetic energy large enough. As they carry a lot of energy, the temperature of the system diminishes after such processes: this phenomenon is known as evaporative cooling and it is widely used to reach extremely low temperatures in most cold atoms experiment. The modelling of two-body elastic losses is a rich topic that we will discuss briefly in the next section.

### 3.3 Evaporation

In our model, we would like to take into account the contribution of two-body elastic losses that provokes evaporative cooling. This problem can be treated using a kinetic theory [14]. We will not derive any result mentioned in this section. All of them are derived in details in a set of lecture notes [15] from C. Cohen-Tannoudji (written in french).
For our system, we consider a 3 dimensional isothermal evaporation in a harmonic trap of depth $U$. The loss rate is

$$\dot{N} = -\Gamma_{ev} N, \quad \Gamma_{ev} = n_0 \sigma_U \frac{\pi e^{-\eta} V_{ev}}{V_e}$$

(3.14)

where $n_0$ is the density at the center of the trap, $\sigma = \frac{8k_B T}{\pi m}$ is the mean quadratic velocity, $\eta = U/k_B T$ is the drap depth, $V_e = (2\pi k_B T/m\omega^2)^{3/2}$ is the effective volume of the sample, i.e. the volume it would take if its density was constant and equal to $n_0$. The evaporative volume $V_{ev}$ is defined as

$$\frac{V_{ev}}{V_e} = 1 - \frac{P(5, \eta)}{P(3, \eta)}$$

(3.15)

The evaporative volume $V_{ev}$ is defined as

$$V_{ev} = (2\pi k_B T/m\omega^2)^{3/2}$$

In order to escape from the trap, an atom must have an energy of at least $U$, or equivalently $\eta k_B T$. In practice, there are also atoms that acquire an energy greater than $\eta k_B T$, and the mean excess energy is denoted $\tilde{\kappa}k_B T$, so that we have the following equation for the energy loss

$$\frac{dE}{dt} = (\eta + \tilde{\kappa})k_B T \frac{dN}{dt}$$

(3.17)

The value of $\kappa$ is known for harmonic traps

$$\tilde{\kappa} = 1 - \frac{P(5, \eta)}{P(3, \eta)} \frac{V_e}{V_{ev}}$$

(3.18)

and using $E = 3Nk_B T$ like we did for three-body losses, we get a differential equation for the temperature

$$\frac{dT}{dt} = \frac{1}{3N} \frac{T}{(\eta + \tilde{\kappa} - 3) \frac{dN}{dt}}$$

(3.19)

### 3.4 Coupled dynamics equations

The model we are going to use extensively in the next chapter to analyse atom number decay and temperature evolution is based on the combination of Eqs. (3.11), (3.13), (3.14) and (3.19), which can be condensed into the following set of equations

$$\begin{cases}
\dot{N} = -\gamma_3 N^3 \frac{T^3}{T^3 + 3} - \gamma_2 e^{-\eta} \frac{V_{ev}}{V_e} N^2 \\
\dot{T} = \frac{T}{3} \left( \gamma_3 \left( \frac{N^2}{T^3 + 3} - \gamma_2 e^{-\eta} \frac{V_{ev}}{V_e} (\eta + \tilde{\kappa} - 3) \frac{N}{T} \right) \right)
\end{cases}$$

(3.20)

where we introduced the constant $\gamma_2 = 8\hbar^2 \pi \beta^3 / \pi k_B U$, and we recall that $\gamma_3 = \lambda_3 (m\beta^2 / 2\sqrt{3} \pi k_B)^3$. These coupled differential equations can be solved numerically (in this work, we used MATHEMATICA).

If the trap depth $\eta$ is large, the evaporative cooling is strongly suppressed by the factor $e^{-\eta}$. In the next chapter, we try to figure out the regimes where it is necessary to take evaporative cooling into account. Of course, if $\eta$ is too big, most of the losses will be caused by two-body elastic scattering, and if we do not take them into account we will seriously
overestimate the three-body loss coefficient $\lambda_3$. We do not have any theoretical estimation for this coefficient yet. Moreover, the temperature exponent $\delta \approx 0.23$ that we want to measure is small. This implies that a great precision is required to extract information from measurements, and justifies the need of analysing this set of equations in order to determine what is the accuracy one should reach to measure the three-body contact. In the following, we wish to construct of robust fit procedure in order to obtain the values of $\delta$ and $\lambda_3$ experimentally.
Chapter 4

Fitting methods

In this last chapter, we present multiple ways to analyse atom number decay and temperature evolution. To begin with, since measurements using $^{40}$K have not been performed yet, we will explain how we produced artificial data and applied artificial noise on them in order to perform an analysis. Then, we will show three distinct methods to analyse the data, each based on non-linear fitting and having pros and cons. Finally, we will apply the three methods on similar data sets, discuss their relative efficiency and domain of application.

4.1 Artificial data and artificial noise

In order to generate artificial data, we solved the differential equations 3.20 numerically. The only parameters required to solve the equations are:

- The mass of $^{40}K$ atoms, $m = 6.636177967 \times 10^{-26}$ kg
- Mean trapping frequency $\omega$
- Initial of number of atoms $N_0$
- Initial temperature $T_0$
- Trap depth $U$, or equivalently, $\eta_0 = U/k_BT_0$
- Temperature dependence of the loss rate $\delta$
- Three-body coefficient $\lambda_3$

The output is a set of points that represents atom number $N(t)$, and temperature $T(t)$, where the time range $[t_{\text{min}}, t_{\text{max}}]$ and the number of points can be chosen. The obtained curves are perfectly smooth, therefore we need to distort them to simulate noise from experimental uncertainty. In this work, we decided to model the noise by performing the following operation on each $N(t)$ and $T(t)$ points

$$N_{\text{noisy}}(t) \rightarrow (1 + p_{N})N(t) \quad T_{\text{noisy}}(t) \rightarrow (1 + p_{T})T(t) \quad \forall t$$  \hspace{1cm} (4.1)

where $p_i$, $i = N, T$ are random number taken from a Gaussian distribution centered in $\mu = 0$ and of standard deviation $\sigma_i$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma^2_i} e^{-\frac{x^2}{2\sigma_i^2}} \quad i = N, T$$  \hspace{1cm} (4.2)

This allows us to tune the noise on the $N(t)$ and $T(t)$ curves independently. Also, since the curves $N(t)$ can vary over more than one order of magnitude, this method ensures that the
noise is always adapted to the scale. In practice, when one performs measurements, the scale of the picture of the cloud is adapted to the size of the cloud, so that the relative error does not depend on the absolute number of atoms or temperature. In order to represent this noise graphically, we used error bars whose sizes are given by the product of the standard error $\sigma_N$ (resp. $\sigma_T$) with the value of the quantity before noise was applied $N(t)$ (resp. $T(t)$).

The Fig. 4.1 represent typical output of the numerical solution for different values of parameters. The smooth curves are the solutions without any noise whereas the dots represent artificial noisy set of data.

Figure 4.1: Left: evolution of atom number. Right: evolution of temperature. The lines represent the numerical solution from the model whereas dots represent artificial data. Parameters are $\delta = 0.23$, $\eta_{\text{init}} = 8$, and $\lambda_3 = 10^{-36}$ S.I. units, except when they are explicitly specified. Initial conditions are $T_0 = 1.0$ mK and $N_0 = 10^5$. The trapping frequency is $\omega = 2\pi \times 112$ Hz. The noise applied to the data is associated with standard errors $\sigma_N = \sigma_T = 0.15$.

The question we want to answer is the following: given multiple noisy sets of data $N_{\text{noisy}}(t)$ and $T_{\text{noisy}}(t)$ generated by $\delta$, $\lambda_3$, $\eta_0$ etc, what is the best strategy to find out what values of $\delta$ and $\lambda_3$ were used to generate the data? And what accuracy can we expect to reach as a function of the intensity of the noise?
4.2 Presentation of various strategies

In this section, we will present three strategies to deduce the values of $\delta$ and $\lambda_3$ a set of $N(t)$ and $T(t)$ curves.

4.2.1 Individual fit (method 1)

This first method consists in taking the value of $\delta = 0.23...$ for granted. The only remaining unknown of the problem is therefore $\lambda_3$. Therefore, one can use the equation (4.20) as a fit model, using the fixed theoretical value for $\delta$ and letting $\lambda_3$ as a free fit parameter. Suppose that we have $p$ sets of curves $(N_i(t), T_i(t))$, $i = 1, 2, ..., p$. We have to imagine that each value of $i$ corresponds to a different experimental realisation: different starting temperature, different number of atoms, different trap depth etc.

From a given set of curves $(N_i(t), T_i(t))$, we use a combined non-linear least-square method to fit the number of atoms and temperature together. There are exactly three fit parameters: $\lambda_3, N_0$ and $T_0$. We can check how much the value of $\lambda_3$ varies between the different set of curves and check whether the results are self-consistent or not. The advantage of this method is that $\lambda_3$ and $\delta$ are not used as free parameters together, so that they can not compensate towards irrelevant values somehow. However, this method can only tell whether choosing $\delta = 0.23...$ is self consistent with observations, but can not predict a value for $\delta$ from scratch. This method is used as a reference in the following to compare the relative accuracy of the other methods.

4.2.2 Fixed temperature fit (method 2)

This method consists in assuming that the temperature is constant during the experiment. This can be realised experimentally by tuning the trap depth in a way such that evaporative cooling and heating due to three-body losses compensate together. Within this hypothesis, the curves $T(t)$ are used to extract the average temperature $\overline{T}$, but are not used in the fitting model. Only the $N(t)$ curves are fitted with the following model

$$\dot{N} = -K_3 N^3 - \gamma_2 e^{-\eta \frac{V_{ev} N^2}{V_e}}$$  \hspace{1cm} (4.3)$$

using $K_3$, $N_0$ and $T_0$ as free parameters. Then, in order to extract $\delta$ and $\lambda_3$, we have to perform this procedure for various average temperature and inspect the dependence of $K_3$ on temperature. Since we have

$$K_3 = \frac{\gamma_3}{T^{3+\delta}}$$  \hspace{1cm} (4.4)$$

the log-log plot of this quantity should be a straight line whose slope is related to the value of $\delta$

$$\ln K_3 = \ln \gamma_3 + (3 + \delta) \ln T$$  \hspace{1cm} (4.5)$$

therefore, once $K_3$ has been measured for enough different temperatures, we can extract the slope with a linear fit. We can also extract the value of $\lambda_3$ from the one of $\ln \gamma_3$.

This method is great in the sense that the number of fitting parameters required for each fit is low, and it allows us to deduce both $\delta$ and $\lambda_3$ from the data without any prior knowledge. However, this method is limited only to $T(t)$ curves that features almost constant temperature. It can not be applied to experimental situations where the temperature varies too much. Its limitations will be characterised later.
4.2.3 Simultaneous fit (method 3)

The last method we used here consists in doing a non-linear combined least-square fitting of all the sets of curves \((N_i(t), T_i(t))\) \(i = 1, ..., p\) together, using the model from Eqs. (3.20). There are a lot of fitting parameters: \(\delta, \lambda_3, N_{0,i}, T_{0,i}, i = 1, ..., p\). The parameters \(\delta\) and \(\lambda_3\) are directly taken to be common to all \((N_i(t), T_i(t))\) sets.

The advantages of this method are obvious: the method does not need any hypothesis about the behaviour of the \(N(t)\) and \(T(t)\) curves, and does not need any prior knowledge of \(\delta\) and \(\lambda_3\). However, there is a drawback. The fact that the method has to deal with a lot of data and handle a lot of fitting parameters makes it highly sensitive to noise. Finding the global minimum with such a method is not an easy task, and convergence is fragile.

An example of the simultaneous fit is shown in Fig. 4.2.

4.3 Comparison of the methods

In this section, we will characterise the efficiency of the previously introduced methods.

Influence of the noise on the prediction To begin with, we can compare the efficiency of the different methods to give a prediction for \(\delta\) and \(\lambda_3\) as a function of the noise. In order to be able to compare the three methods, we place ourselves in a situation where \(T\) is nearly constant (it does not vary more than 2\% from its mean value). Therefore, we can compare methods 1, 2 and 3 altogether. In order to see the influence of the noise on the 3 methods we applied the following procedure:

- Generate 11 sets of curves \((N(t), T(t))\), each featuring different but nearly constant temperature and containing 11 data points. The noise applied on these data is identical for every set and characterised by \(\sigma_N\) and \(\sigma_T\).
- The values of \(\delta\) and \(\lambda_3\) are computed using the three different methods.
- We start again the procedure 10 times to finally obtain the averaged predicted value for \(\delta\) and \(\lambda_3\), and the associated standard deviation.

After performing the procedure several times for various \(\sigma_N\) and \(\sigma_T\), we obtain the influence of the noise on the prediction for \(\delta\) and \(\lambda_3\) for all the methods. For instance, the Figure 4.3 represents the influence of \(\sigma_N\) at fixed \(\sigma_T\).

From what we can see, it seems that in this constant temperature context, each method produces equivalent results at low noise. We note however the instability of the simultaneous fit when the noise increases too much. The fact that there is no obvious difference between method 1 and 2 is justified by the fact that these methods should be equivalent at nearly constant temperature.

Robustness of method 2 when temperature varies It is interesting to determine precisely when the method 2 breaks down if the temperature starts to vary significantly in a data set. For that purpose, we define the quantity \(\Delta_T\) that measures the relative variation of temperature from multiple sets of data points \((N_i(t), T_i(t)), i = 1, ..., p\)

\[
\Delta_T = 1 - \frac{1}{p} \sum_{i=1}^{p} \frac{T_{i,\text{initial}} - T_{i,\text{final}}}{\overline{T}_i}
\]  (4.6)

where \(T_{i,\text{initial}}, T_{i,\text{final}}\) and \(\overline{T}_i\) represent the initial temperature, the final temperature and the average temperature associated with the set of temperature data points \(i\), respectively. Now, using a procedure similar to the one of the previous paragraph, we can plot how the
Figure 4.2: Input and output of the combined fit method. The dots with error bars correspond to artificial data. The blue curve corresponds to an initial guess for the least square optimization. The red curve corresponds to the optimized parameters, including $\delta$ and $\lambda_3$ common to each curve. We used 4 sets of curves $(N(t), T(t))$, each containing 11 points. In this example, the guess parameters were $\lambda_{3,g} = 0.1 \times 10^{-37}$ S.I. and $\delta_g = 0.1$ where the parameters used to generate the data were $\lambda_3 = 10^{-37}$ S.I. and $\delta = 0.23$. The optimization returned $\delta_{opt} = 0.235 \pm 0.07$ and $\lambda_{3, opt} = (1.16 \pm 0.8) \times 10^{-37}$ S.I. The noise used here is $\sigma_N = 0.15$ and $\sigma_T = 0.04$.

Predictions on $\delta$ and $\lambda_3$ vary when $\Delta T$ changes, as shown in Figure 4.3. We observe that the error bars increase when $|\Delta T|$ increases, and also that the average value for $\delta$ and $\lambda_3$ deviate from the target value when $\Delta T$ increases while being positive, corresponding to situation when temperature decreases over time.
Figure 4.3: Influence of the atom number noise $\sigma_N$ at fixed temperature noise $\sigma_T$ on the prediction for $\delta$ and $\lambda_3$ for the 3 methods. The dashed line represents the target value. Error bars represent the standard deviation.

**Can we get rid of trapping potential?** If we want to take evaporative cooling into account in our model, this requires the knowledge of the trap depth $U$. However, if we could somehow minimize the effects of two-body losses, we could get rid of the knowledge of the precise value of $U$. This would remove all uncertainties due to an imprecise knowledge of $U$, and the data analysis would not require any model for evaporation. In practice, if we set the initial relative trap depth $\eta$ to a high value, the trap is too deep and two-body losses will not occur. This is justified by the factor $e^{-\eta}$ that becomes really small in Eqs. (3.20). On the other hand, if only three-body losses occur in the system, we will certainly observe heating, the temperature will increase and methods 1 and 2 will not be applicable. The method 3 however, remains useful. We can test the influence of evaporative cooling by slightly modifying the model. We can perform the fit without taking evaporative cooling into account. Thus, we use the following equations for the fit

$$
\begin{align*}
\dot{N} &= -\gamma_3 \frac{N^3}{T^{3+3}} \\
\dot{T} &= \frac{T}{3} \gamma_3 \left(1 + \frac{\delta}{3}\right) \frac{N^2}{T^{3+3}}
\end{align*}
$$

(4.7)

We can determine how the predictions on $\delta$ and $\lambda_3$ are modified when evaporative cooling is not taken into account within the fit model. From Figure 4.5, we can establish the domain of validity of Eqs. (4.7).
Figure 4.4: Evolution of the prediction on $\delta$ and $\lambda_3$ using method 1 and 2. A constant noise $\sigma_N = \sigma_T = 0.06$ was used. The dashed line represents the target value. Each point corresponds to an average of 30 realisations, and the error bars correspond to the associated standard deviation.

Figure 4.5: Evolution of the predictions on $\delta$ and $\lambda_3$ using method 3 without evaporative cooling as a function of the trap depth. This graph allows to benchmark the relevance of forgetting the evaporative cooling in the model. A constant noise of $\sigma_N = 0.10$ and $\sigma_T = 0.04$ was used, each point corresponds to an average over 50 realisations, and error bars represent to the associated standard deviation. When $\eta \gtrsim 20$, evaporative cooling is unimportant in this situation.
Conclusion

During this work, we have shown that three-body losses could be used to probe many-body correlations. The measurement of inelastic losses in a weakly coupled Bose-Fermi gas coincide with the theoretical predictions for the two-body contact $C_2$ \[1\]. This problem is related to two-body correlations because during three-body losses, due to weak interactions, the statistics of the bosonic impurity is decoupled from the one of the fermions.

After reviewing fundamentals on the two-body and the three-body problem, we showed how we can generalize this approach and define analogously the three-body contact $C_3$, related to the three-body correlations in the many-body problem. If we consider a pure Bose gas or a pure Fermi gas, we expect the probability that a three-body recombination occurs to be proportional to $C_3$. For a Bose gas and at unitarity, the three-body loss coefficient has been measured and the result is consistent with theoretical predictions \[13\]. However, for a Fermi gas at unitarity, there is currently no theoretical prediction and experimental determination of $C_3$. However, as we mentioned here, scaling relations, which successfully describe the losses in the weakly coupled Bose-Fermi gas, tell us how the decay rate should behave. This fact motivates the measurement of three-body losses in a Fermi gas using ultracold atoms of $^{40}K$ that will be performed in the near future in our group.

In the non degenerate regime, we expect the dependence of the decay rate on temperature to be very small, and this justifies the development of a robust data analysis procedure in order to be able to characterise the small exponent $\delta$, as well as the proportionality coefficient $\lambda_3$. In the last chapter, we presented different methods to analyse $N(t)$ and $T(t)$ curves in order to extract the desired quantities. According to the experimental data that will be collected, we will be able to choose the strategy that is the most relevant. We also showed how computer simulations could help us to determine the error associated with the predictions.

This work will pave the way to a better understanding of the effect of three-body interactions that arise for instance in nuclear matter \[6\]. Indeed, characterizing the three-body contact would be useful to tackle three-body interactions perturbatively. Consider the following Hamiltonian

$$\mathcal{H}_{2b} + \epsilon \int d^3r_1d^3r_2d^3r_3 V(r_1, r_2, r_3) \hat{\Psi}_1^\dagger(r_1) \hat{\Psi}_2^\dagger(r_2) \hat{\Psi}_3^\dagger(r_3) \hat{\Psi}_2(r_2) \hat{\Psi}_1(r_1)$$ \hspace{1cm} (4.8)

Here $\mathcal{H}_{2b}$ is the usual many-body Hamiltonian with two-body interactions. If we consider that we know how to solve $\mathcal{H}_{2b}$ (which indeed, is not a trivial problem at all), and if $\epsilon$ is a small parameter, we can approach the problem perturbatively. This requires the knowledge of the three-body correlation function $\langle \hat{\Psi}_1^\dagger(r_1) \hat{\Psi}_2^\dagger(r_2) \hat{\Psi}_3^\dagger(r_3) \hat{\Psi}_2(r_2) \hat{\Psi}_1(r_1) \rangle$ where the mean value is taken in the many-body ground state of $\mathcal{H}_{2b}$. If the three-body potential $V(r_1, r_2, r_3)$ is restricted to short-range interactions, this quantity is determined by the knowledge of the three-body contact $C_3$. 

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Appendix

Heating effect caused by three-body recombination

In a homogeneous system, the atom number decay is given by

$$\frac{dn}{dt} = -L_3 n^3$$  \hspace{1cm} (9)

where the three-body coefficient \( L_3 \) describes the rate at which atoms are lost due to three-body recombination. It is expressed as an average over all possible incoming momenta

$$L_3 = \langle L_3(p_1, p_2, p_3) \rangle = \frac{1}{Z} \int d^3p_1 d^3p_2 d^3p_3 L_3(p_1, p_2, p_3) f_\uparrow(p_1) f_\uparrow(p_2) f_\downarrow(p_3)$$  \hspace{1cm} (10)

where \( L_3(p_1, p_2, p_3) \) represents the rate at which atoms are lost because of a recombination event with incoming particles momenta \( p_1, p_2 \) and \( p_3 \). We place ourselves in a limit where the distribution functions are spin-independent Maxwell distributions. The normalization \( Z \) is

$$Z = \int d^3p_1 d^3p_2 d^3p_3 f_\uparrow(p_1) f_\uparrow(p_2) f_\downarrow(p_3)$$  \hspace{1cm} (11)

We use Jacobi coordinates \((P, \Pi_1, \Pi_2)\) to simplify the problem thanks to translational invariance

$$L_3(p_1, p_2, p_3) = L_3(\Pi_1, \Pi_2)$$  \hspace{1cm} (12)

because \( P \) is the center of mass momentum, and \( \Pi_1 \) and \( \Pi_2 \) are two relatives momenta.

$$\begin{cases} p_1 = \frac{1}{3} P - \frac{1}{a} \Pi_1 - \frac{a}{2} \Pi_2 \\ p_2 = \frac{1}{3} P + \frac{1}{a} \Pi_1 - \frac{a}{2} \Pi_2 \\ p_3 = \frac{1}{3} P + a \Pi_2 \end{cases}$$  \hspace{1cm} (13)

here, \( a^4 = 4/3 \) is chosen in such a way that the total kinetic energy has the symmetric following form

$$\frac{h^2(k_1^2 + k_2^2 + k_3^2)}{2m} = \frac{P^2}{2M} + \frac{\Pi^2}{2\mu}$$  \hspace{1cm} (14)

where \( \Pi^2 = \Pi_1^2 + \Pi_2^2 \), also \( M = 3m \) is the total mass and \( \mu = m/\sqrt{2} \) is the reduced mass. Therefore, in the high temperature limit where we can assume Boltzmann distributions for \( f_\sigma(p_i) \), we have

$$f_\uparrow(p_1) f_\uparrow(p_2) f_\downarrow(p_3) = e^{-\beta P^2/2M} e^{-\beta \Pi^2/2\mu}$$  \hspace{1cm} (15)

In hyper-spherical coordinates, the problem simplifies even further

$$L_3(p_1, p_2, p_3) = |\phi_{\ell, n}(\Omega)|^2 \tilde{L}_3(\Pi)$$  \hspace{1cm} (16)
With \( \ell = 1, n = 0 \) corresponding to the channel with the lowest \( s_{\ell,n} \) exponent. We can therefore integrate using hyper-spherical, using that

\[
d^3p_1 d^3p_2 d^3p_3 = \left( \frac{3}{2} \right)^{3/2} d^3P \Pi^5 d\Pi d^5\Omega
\]

The angular integration is straightforward as

\[
\int d^5\Omega |\phi_{\ell,n}(\Omega)|^2 = 1.
\]

We can perform the integration on the total momentum as well

\[
\int d^3P e^{-\beta P^2/2M} = 4\pi \int_0^{+\infty} dP P^2 e^{-\beta P^2/2M} = (2\pi k_B T M)^{3/2}
\]

but this contribution will cancel anyway Therefore,

\[
L_3 = \frac{\int_0^{+\infty} d\Pi \Pi^5 e^{-\beta \Pi^2/2\mu} \tilde{F}(\Pi)}{\int_0^{+\infty} d\Pi \Pi^5 e^{-\beta \Pi^2/2\mu}}
\]

Of course, we need the expression of \( \tilde{L}_3(\Pi) \) to keep going, but it arises from the complicated three-body physics.

**ASSUMPTION** Let us consider \( \tilde{L}_3(\Pi) = \kappa \Pi^\alpha \) where we assume \( \alpha \) not to be necessarily an integer. Then we have to use the following result

\[
\int_0^{+\infty} d\Pi \Pi^p e^{-a\Pi^2} = \frac{1}{2} a^{-1/2-p/2} \Gamma \left( \frac{1+p}{2} \right) \quad \text{if} \quad p > -1, \ a > 0
\]

where \( \Gamma(z) \) is the usual Gamma function

\[
\Gamma(z) = \int_0^{\infty} dx x^{z-1} e^{-x}
\]

Therefore, we have

\[
L_3 = \kappa \frac{(2\mu k_B T)^{3+\alpha/2} \Gamma \left( \frac{6+\alpha}{2} \right)}{2 (2\mu k_B T)^{\alpha/2}} = \kappa \frac{\alpha}{2} \Gamma \left( \frac{6+\alpha}{2} \right)
\]

The integration was performed provided

\[
5 + \alpha > -1 \quad \Rightarrow \quad \alpha > -6
\]

Now, suppose that we put the system in a harmonic potential. Therefore, the density is no longer constant and depends on the temperature as well. In order to obtain the variation of the total atom number, we have to perform the spatial integration

\[
\frac{dN_3}{dt} = -L_3 \int d^3r n^3(r) = -L_3 N \langle n^2 \rangle
\]

the density is given by

\[
n(r) = n(0) e^{-\beta U(r)}
\]

where \( U(r) = U(x,y,z) = \frac{1}{2} m(x^2 + y^2 + z^2) \) is the trapping potential. The density at the center of the potential can be expressed as

\[
n(0) = \frac{N}{\lambda_{th}^3} \zeta
\]
where $\lambda_{th}$ is the thermal wavelength

$$\lambda_{th} = \sqrt{\frac{2\pi \hbar^2}{mk_B T}}$$

and $\zeta$ is the partition function

$$\zeta = \frac{1}{(2\pi\hbar)^3} \int d^3r d^3p \ e^{-\beta(p^2/2m + U(r))} = \frac{1}{(2\pi\hbar)^3} \left( \frac{2\pi k_B T}{\omega} \right)^3$$

with $\omega = (\omega_x \omega_y \omega_z)^{1/3}$. Therefore, we can compute

$$\langle n^2 \rangle = N^2 \left( \frac{m \omega^2}{2\sqrt{3\pi k_B T}} \right)^3$$

From this we deduce that

$$\hat{N}_{3b} \sim T^{\alpha/2-3}$$

which allows us to determine that $\alpha = -2\delta$. Notice that the condition $\alpha > -6$ is satisfied since we expect to have $\delta \approx 0.23$. Now, we want to compute the change energy lost by the system due to three-body recombination per lost atom.

$$\frac{E_{3b}}{\hat{N}_{3b}} = \frac{-\int d^3r d^3p_1 d^3p_2 d^3p_3 n^3(r) L_3(p_1, p_2, p_3) f_1(p_1) f_1(p_2) f_1(p_3) \left( \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{p_3^2}{2m} + 3U(r) \right) - \int d^3r d^3p_1 d^3p_2 d^3p_3 n^3(r) 3L_3(p_1, p_2, p_3) f_1(p_1) f_1(p_2) f_1(p_3) \left( \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{p_3^2}{2m} + 3U(r) \right)}{\int d^3r d^3p_1 d^3p_2 d^3p_3 n^3(r) L_3(p_1, p_2, p_3) f_1(p_1) f_1(p_2) f_1(p_3) \left( \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{p_3^2}{2m} + 3U(r) \right)}$$

The integral over momenta in the numerator can be transformed into Jacobi coordinates

$$\left( \frac{3}{2} \right)^{3/2} \int d^3\mathbf{P} \Pi^5 \Pi L_3(\Pi) e^{-\beta P^2/2m} e^{-\beta\Pi^2/2\mu} \left( \frac{P^2}{2M} + \frac{\Pi^2}{2\mu} + 3U(r) \right)$$

We can evaluate the three terms one by one. Let us start by the center of mass term

$$\frac{\dot{E}_P}{\hat{N}_{3b}} = \frac{-\int d^3\mathbf{P} \Pi^2 \Pi e^{-\beta P^2/2M}}{3 \int d^3\mathbf{P} \Pi^2 e^{-\beta P^2/2M}} = \frac{1}{6} \left( \frac{2\pi M}{2M} \right)^{3/2} \left( \frac{k_B T}{2M} \right)^{5/2} = \frac{1}{3} k_B T$$

Then, we can perform the hyper-momentum integration

$$\frac{\dot{E}_{11}}{\hat{N}_{3b}} = \frac{1}{6 \mu} \left( \frac{2\pi k_B T}{2\mu} \right)^{4+\alpha/2} \Gamma \left( \frac{8+\alpha}{2} \right) \Gamma \left( \frac{6+\alpha}{2} \right) = \frac{1}{3} k_B T \frac{\Gamma \left( \frac{8+\alpha}{2} \right)}{\Gamma \left( \frac{6+\alpha}{2} \right)}$$

provided $\alpha > -8$, which is satisfied since we have $\alpha > -6$. This result can be simplified using the property

$$\frac{\Gamma(p+1)}{\Gamma(p)} = p$$

so we end up with

$$\frac{\dot{E}_{11}}{\hat{N}_{3b}} = \left( 1 + \frac{\alpha}{6} \right) k_B T = \left( 1 - \frac{\delta}{3} \right) k_B T$$

It remains to compute the potential energy term

$$\frac{\dot{E}_U}{\hat{N}_{3b}} = \frac{3 \int d^3r n^3(r) U(r)}{3 \int d^3r n^3(r)} = \frac{\int d^3r U(r) e^{-3\beta U(r)}}{\int d^3r e^{-3\beta U(r)}}$$
specifying the expression of $U(r)$, we notice that all 3 terms play an equivalent role due to normalization, therefore we only have left to compute one integral

$$\frac{\dot{E}_U}{N_{3b}} = 3\int \frac{dx}{x^2} \frac{1}{\omega_x} e^{-3\beta \frac{1}{2} m \omega_x x^2} = \frac{3}{2} m \omega_x \left( \frac{2 k_B T}{3 m \omega_x} \right)^{3/2} \frac{\Gamma \left( \frac{3}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} = \frac{1}{2} k_B T \quad (38)$$

finally we just have to add all the contributions $\dot{E}_{3b} = \dot{E}_P + \dot{E}_\Pi + \dot{E}_U$ to obtain

$$\frac{\dot{E}_{3b}}{N_{3b}} = k_B T \left( 2 - \frac{\delta}{3} \right) \quad (39)$$
Bibliography


