Atoms and photons

Chapter 3

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Outline

1. Planck 1900

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2. Field eigenmodes

Outline

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3. Field quantization

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2. Field eigenmodes
3. Field quantization
4. Field quantum states

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2. Field eigenmodes
3. Field quantization
4. Field quantum states
5. Beamsplitter

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2. Field eigenmodes
3. Field quantization
4. Field quantum states
5. Beamsplitter
6. Field relaxation

The Blackbody problem

Emission by a small hole in a heated oven. What is known at Planck's time.

- The radiation is universal
- Stefan's law

$$\mathcal{P} = \sigma S T^4 \tag{1}$$

  where $\sigma = 5.67\,10^{-8}$ W/m^{2}K^4
- Lambert's law

$$d\mathcal{P} = LS\cos\theta\, d\Omega \tag{2}$$

  where the luminance L is related to the total density of energy in the oven $u = \int u_\nu\, d\nu$, by:

$$L = \frac{cu}{4\pi} \tag{3}$$

$$\mathcal{P} = \frac{cSu}{4} \tag{4}$$

  and

$$u = \frac{4}{c}\sigma T^4 \tag{5}$$

The Blackbody problem

Emission by a small hole in a heated oven. What is known at Planck's time?

- Wien's displacement law

$$u_\nu = \nu^3 f\left(\frac{\nu}{T}\right) \tag{6}$$

- Wien's phenomenological model

$$u_v = \alpha\nu^3 e^{-\gamma\nu/T} \ , \tag{7}$$

- And many precise measurements of the spectrum (pyrometry).

The Blackbody problem

Counting the modes

Assume a rectangular volume for the oven, with periodic boundary conditions. Support only plane waves with $\mathbf{k} = (k_x, k_y, k_z)$ so that

$$k_x = \frac{2\pi}{L_x} n_x \tag{8}$$

where $n_{x,y,z}$ is a set of three positive or negative integers. Two orthogonal polarizations for each set of integers. Energies of all these 'modes' add up independently (detailed justification later).

N_ν the total number of modes $k < 2\pi\nu/c$. Number of modes per unit volume between ν and $\nu + d\nu$: $\rho_\nu \, d\nu$

$$\rho_\nu = \frac{1}{\mathcal{V}} \frac{dN_\nu}{d\nu} \tag{9}$$

The Blackbody problem

Counting the modes

Counting the modes with a frequency lower than ν amounts to counting twice the number of points with integer coordinates in a sphere of radius $2\pi\nu/c$:

$$N_\nu = 2\frac{\frac{4\pi}{3}\left(\frac{2\pi\nu}{c}\right)^2}{\frac{8\pi^3}{\mathcal{V}}} = \frac{8\pi}{3}\frac{\nu^3}{c^3}\mathcal{V} \tag{10}$$

where $\mathcal{V}$ is the box volume. Hence

$$\rho_\nu = \frac{8\pi}{c^3}\nu^2 \tag{11}$$

The Blackbody problem

Rayleigh Jeans argument

Attribute the average thermal energy $k_b T$ to each mode

$$u_\nu = k_b T \rho_\nu \tag{12}$$

- Fits with observation at low frequency
- Absurd at high frequencies: divergence of the spectrum and infinite power

Classical statistical physics fails at explaining the blackbody radiation !

The Blackbody problem

Planck's argument

The light quantum

Planck's hypothesis

The exchanges of energy between field and matter occur as multiples of a fundamental quantum

$$h\nu \tag{13}$$

where h is a 'Hilfeconstant'. Hence $E = nh\nu$.

Average energy per mode (standard statistical physics)

$$\overline{E} = h\nu \frac{\sum_{n=0}^{\infty} n e^{-nh\nu/k_b T}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_b T}} \tag{14}$$

The Blackbody problem

Planck's argument

With $\beta = 1/k_b T$ and $\chi = \beta h\nu$, we note that
$\sum \exp(-\chi n) = 1/[1 - \exp(-\chi)]$ and
$\sum n \exp(-\chi n) = -(d/d\chi)1/[1 - \exp(-\chi)] = \exp(-\chi)/[1 - \exp(-\chi)]^2$

$$\overline{E} = h\nu\overline{n} = h\nu\frac{1}{e^{\chi} - 1} \tag{15}$$

We finally get the Planck's law:

$$u_\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_b T} - 1} \tag{16}$$

In excellent agreement with experiments if

$$h = 6.62\,10^{-34}\ \text{J/s} \tag{17}$$

The Blackbody problem

Limits

- For small frequencies: Rayleigh Jeans

$$u_\nu = \frac{8\pi\nu^2}{c^3} k_b T \tag{18}$$

the classical predictions without field quantization (many photons per mode).

- For large frequencies: phenomenological Wien's law

$$u_\nu = \frac{8\pi h\nu^3}{c^3} e^{-h\nu/k_b T} \tag{19}$$

- Explicit expression of Stefan's constant

$$\sigma = \frac{2\pi^5}{15} \frac{k_b^4}{c^2 h^3} \tag{20}$$

The Blackbody problem

Einstein 1905

A more solid justification of the heuristic Plank's hypothesis. Starting point

$$u_\nu = \alpha\nu^3 e^{-h\nu/k_b T} = \alpha\nu^3 e^{-\gamma\nu T} \quad (21)$$

with $\gamma = h/k_b$. This leads by a simple inversion to:

$$T = -\frac{\gamma\nu}{\ln u_\nu/\alpha\nu^3} \quad (22)$$

Density of entropy s, $ds/du = 1/T$ and, by integration over u

$$\begin{aligned} s &= -\int_0^\infty du' \, \frac{\ln u'/\alpha\nu^3}{\gamma\nu} \\ &= -\frac{u}{\gamma\nu}\left[\ln\frac{u}{\alpha\nu^3} - 1\right] \end{aligned} \quad (23)$$

The Blackbody problem

Einstein 1905

Total entropy in volume $\mathcal{V}$, $S = s\mathcal{V}$, and total energy $E = u\mathcal{V}$ linked by

$$S = -\frac{E}{\gamma\nu}\left[\ln\frac{E}{\mathcal{V}\alpha\nu^3} - 1\right] \tag{24}$$

S_0 the entropy for the volume $\mathcal{V}_0$

$$S - S_0 = \frac{E}{\gamma\nu}\ln\frac{\mathcal{V}}{\mathcal{V}_0} \tag{25}$$

Compare to the entropy variation of a perfect gas in an isothermal compression

$$S - S_0 = k_b N \ln\frac{\mathcal{V}}{\mathcal{V}_0} \tag{26}$$

where N is the total number of particles. $Nk_b = Ek_b/h\nu$ and $E/N = h\nu$.

Objective

To quantify the field, we must identify a set of orthogonal modes, the relevant dynamical variables and quantify them according to the 'canonical' quantization procedure. The main technical difficulty in field quantization is thus a classical electromagnetism calculation.

Eigenmodes

Positive frequency fields

Time Fourier transform of electric field

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{\mathbf{E}}(\mathbf{r},\omega) e^{-i\omega t}\, d\omega \tag{27}$$

Since $\mathbf{E}$ is a real field,

$$\widetilde{\mathbf{E}}^*(\mathbf{r},\omega) = \widetilde{\mathbf{E}}(\mathbf{r},-\omega) \tag{28}$$

Define the ‘positive frequency field’

$$\mathbf{E}^+(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \widetilde{\mathbf{E}}(\mathbf{r},\omega) e^{-i\omega t}\, d\omega \tag{29}$$

and the ‘negative frequency field’

$$\mathbf{E}^-(\mathbf{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \widetilde{\mathbf{E}}(\mathbf{r},\omega) e^{-i\omega t}\, d\omega = \left(\mathbf{E}^+(\mathbf{r},t)\right)^* \tag{30}$$

Hence

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^+(\mathbf{r},t) + \mathbf{E}^-(\mathbf{r},t) \tag{31}$$

Eigenmodes

Eigenmodes basis

'Box' of limiting conditions with a total volume $\mathcal{V}$. Orthogonal basis for the solutions of Maxwell equations (a Hilbert space)

$$\mathbf{f}_\ell(\mathbf{r})e^{-i\omega_\ell t} \tag{32}$$

where the dimensionless amplitude $\mathbf{f}_\ell$ is divergence-free and obeys the Helmholtz equation:

$$\Delta\mathbf{f}_\ell + \frac{\omega_\ell^2}{c^2}\mathbf{f}_\ell = 0 \tag{33}$$

Orthogonality:

$$\int_{\mathcal{V}} d^3\mathbf{r}\,\mathbf{f}_\ell^*(\mathbf{r})\cdot\mathbf{f}_{\ell'}(\mathbf{r}) = \delta_{\ell,\ell'}\mathcal{V} \tag{34}$$

Normalization:

$$\int_{\mathcal{V}} d^3\mathbf{r}\,|\mathbf{f}_\ell(\mathbf{r})|^2 = \mathcal{V} \tag{35}$$

Eigenmodes

Eigenmodes basis

Expand the positive frequency field on this basis

$$\mathbf{E}^{+}(\mathbf{r},t)=\sum_{\ell}\mathcal{E}_{\ell}(t)\mathbf{f}_{\ell}(\mathbf{r}) \tag{36}$$

where

$$\mathcal{E}_{\ell}(t)=\frac{1}{\mathcal{V}}\int\mathbf{E}^{+}(\mathbf{r},t)\cdot\mathbf{f}_{\ell}^{*}(\mathbf{r})\,d^{3}\mathbf{r} \tag{37}$$

The amplitude is obviously a harmonic function of time

$$\mathcal{E}_{\ell}(t)=\mathcal{E}_{\ell}(0)e^{-i\omega_{\ell}t} \tag{38}$$

Finally

$$\mathbf{E}^{+}(\mathbf{r},t)=\sum_{\ell}\mathcal{E}_{\ell}(0)e^{-i\omega_{\ell}t}\mathbf{f}_{\ell}(\mathbf{r}) \tag{39}$$

Eigenmodes

Plane-wave basis

- A simple basis for a rectangular box and periodic boundaries.
- Set of plane waves with $\mathbf{k_n} = (k_x, k_y, k_z) = (n_x 2\pi/L_x, n_y 2\pi/L_y, n_z 2\pi/L_z)$, where the ns are positive or negative.
- For each $\mathbf{n} = (n_x, n_y, n_z)$, two orthogonal linear polarizations $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, perpendicular to $\mathbf{k}$: $\boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2 = \mathbf{u_k}$.
- Basis

$$\mathbf{f}_\ell(\mathbf{r}) = \boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \tag{40}$$

  with $\ell = (n_x, n_y, n_z, \epsilon)$
- Circular polarization basis

$$\boldsymbol{\epsilon}_\pm = \frac{\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2}{\sqrt{2}} \tag{41}$$

$$\boldsymbol{\epsilon}_+ \times \boldsymbol{\epsilon}_- = -i\mathbf{u_k} \tag{42}$$

Eigenmodes

Mode basis change

Two sets of modes $\mathbf{f}_\ell$ and $\mathbf{g}_p$ checking the same limiting conditions

$$\mathbf{f}_\ell = \sum_p U_{\ell p}\mathbf{g}_p \ . \tag{43}$$

where $U_{\ell p}$ connects only modes with the same frequency.

$$U_{\ell p} = \frac{1}{\mathcal{V}} \int \mathbf{f}_\ell \cdot \mathbf{g}_p^* \, d^3\mathbf{r} \tag{44}$$

Eigenmodes

Mode basis change

Check that U is unitary

$$\delta_{\ell,\ell'} = \frac{1}{\mathcal{V}} \int \mathbf{f}_\ell^* \cdot \mathbf{f}_{\ell'}\, d^3\mathbf{r} = \sum_{p,p'} U_{\ell p}^* U_{\ell' p'} \frac{1}{\mathcal{V}} \int \mathbf{g}_p^* \cdot \mathbf{g}_{p'}\, d^3\mathbf{r} \tag{45}$$

Using the orthonormality of $\mathbf{g}$:

$$\delta_{\ell,\ell'} = \sum_p U_{\ell p}^* U_{\ell' p} = \sum_p U_{\ell' p} U_{p\ell}^\dagger \tag{46}$$

and hence $\mathbb{1} = UU^\dagger$

Normal variables

Potential vector

Choose a simple set of dynamical variables. The potential vector $\mathbf{A}$ is divergence-free in the Coulomb gauge and $\mathbf{E} = -\partial\mathbf{A}/\partial t$. Can be thus expanded on the same basis as $\mathbf{E}$

$$\mathbf{A}^{+}(\mathbf{r}, t) = \sum_{\ell} \mathcal{A}_{\ell}(t)\mathbf{f}_{\ell}(\mathbf{r}) \tag{47}$$

Choose the $\mathcal{A}(t)$ (harmonic functions of time) as the normal variables and separate real and imaginary parts

$$\mathcal{A}_{\ell}(t) = \mathcal{A}_{\ell}(0)e^{-i\omega t} = x_{\ell}(t) + ip_{\ell}(t) \tag{48}$$

Normal variables

All fields

From $\mathbf{E}^+ = -\partial \mathbf{A}^+/\partial t$

$$\mathcal{E}_\ell(t) = -\frac{d\mathcal{A}_\ell}{dt} = i\omega_\ell \mathcal{A}_\ell \tag{49}$$

and hence

$$\mathbf{E}^+(\mathbf{r}, t) = \sum_\ell i\omega_\ell \mathcal{A}_\ell(t)\mathbf{f}_\ell(\mathbf{r}) \tag{50}$$

Magnetic field:

$$\mathbf{B}^+(\mathbf{r}, t) = \sum_\ell \mathcal{A}_\ell(t)\mathbf{h}_\ell(\mathbf{r}) \tag{51}$$

where

$$\mathbf{h}_\ell(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{f}_\ell(\mathbf{r}) \tag{52}$$

Field energy

The total field energy

$$H = \frac{\epsilon_0}{2}\int E^2 + \frac{1}{2\mu_0}\int B^2 \tag{53}$$

must be written in terms of real fields

$$\mathbf{E} = 2\mathrm{Re}\,\mathbf{E}^+ = 2\mathrm{Re}\sum_\ell i\omega_\ell \mathcal{A}_\ell \mathbf{f}_\ell \tag{54}$$

Taking into account the mode orthogonality

$$H = \sum_\ell H_\ell \tag{55}$$

Remains to evaluate energy of one given mode. Drop index ℓ for the time being.

Field energy

Electric energy

Real field

$$\mathbf{E} = i\omega\left[\mathcal{A}\mathbf{f} - \mathcal{A}^*\mathbf{f}^*\right] \tag{56}$$

or

$$\mathbf{E} = -2\omega\left[x\mathbf{f}'' + p\mathbf{f}'\right] \tag{57}$$

with

$$\mathbf{f} = \mathbf{f}' + i\mathbf{f}'' \tag{58}$$

$$H_e = 2\omega^2\epsilon_0\left[x^2\int(\mathbf{f}'')^2 + p^2\int(\mathbf{f}')^2 + 2xp\int\mathbf{f}'\cdot\mathbf{f}''\right] \tag{59}$$

Field energy

Magnetic energy

With

$$\mathbf{B} = \mathcal{A}\mathbf{h} + \mathcal{A}^*\mathbf{h}^* = 2x\mathbf{h}' - 2p\mathbf{h}'' \qquad (60)$$

we get

$$H_b = \frac{2}{\mu_0}\left[x^2\int(\mathbf{h}')^2 + p^2\int(\mathbf{h}'')^2 - 2xp\int\mathbf{h}'\cdot\mathbf{h}''\right] \qquad (61)$$

Similar, but not obviously equal, to the electric energy.

Field energy

Comparing the energies

Let us start with the integral of $(\mathbf{h}')^2$, with $\mathbf{h} = \boldsymbol{\nabla} \times \mathbf{f}$. Using

$$\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b}) \tag{62}$$

we can write

$$\boldsymbol{\nabla} \cdot [\mathbf{f}' \times (\boldsymbol{\nabla} \times \mathbf{f}')] = (\boldsymbol{\nabla} \times \mathbf{f}')^2 - \mathbf{f}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{f}') \tag{63}$$

Using that these fields are divergence-free and with Helmoltz equation:

$$\boldsymbol{\nabla} \cdot [\mathbf{f}' \times (\boldsymbol{\nabla} \times \mathbf{f}')] = (\mathbf{h}')^2 - \frac{\omega^2}{c^2}(\mathbf{f}')^2 \tag{64}$$

Integrating over space:

$$\int (\mathbf{h}')^2 = \frac{\omega^2}{c^2} \int (\mathbf{f}')^2 \tag{65}$$

Similarly

$$\int (\mathbf{h}'')^2 = \frac{\omega^2}{c^2} \int (\mathbf{f}'')^2 \tag{66}$$

Field energy

Comparing the energies

Let us examine is $\int \mathbf{h}' \cdot \mathbf{h}''$. With

$$\boldsymbol{\nabla} \cdot [\mathbf{f}' \times (\boldsymbol{\nabla} \times \mathbf{f}'')] = (\boldsymbol{\nabla} \times \mathbf{f}) \cdot (\boldsymbol{\nabla} \times \mathbf{f}'') - \mathbf{f}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{f}'') \tag{67}$$

we get

$$\int \mathbf{h}' \cdot \mathbf{h}'' = \frac{\omega^2}{c^2} \int \mathbf{f}' \cdot \mathbf{f}'' \tag{68}$$

Hence

$$H_b = 2\omega^2 \epsilon_0 \left[x^2 \int (\mathbf{f}')^2 + p^2 \int (\mathbf{f}'')^2 - 2xp \int \mathbf{f}' \cdot \mathbf{f}'' \right] \tag{69}$$

Using

$$\int (\mathbf{f}')^2 + \int (\mathbf{f}'')^2 = \mathcal{V} \tag{70}$$

we get finally

$$H = 2\omega^2 \epsilon_0 \mathcal{V} \left[x^2 + p^2 \right] \tag{71}$$

Field energy

Total field energy

The total energy of the radiation field is thus:

$$H = \sum_\ell H_\ell = \sum_\ell 2\omega_\ell^2 \epsilon_0 \mathcal{V} \left[x_\ell^2 + p_\ell^2 \right] \quad (72)$$

A collection of independent harmonic oscillators.

Field energy

Canonical variables

- Need canonically conjugate variables for quantization: x_c and p_c such that

$$\frac{dx_c}{dt} = \frac{\partial H}{\partial p_c} \qquad \text{and} \qquad \frac{dp_c}{dt} = -\frac{\partial H}{\partial x_c} \tag{73}$$

- x and p are not canonical, since

$$\frac{dx}{dt} = \omega p \neq \frac{\partial H}{\partial p} = 4\omega^2\epsilon_0\mathcal{V}p \tag{74}$$

- Canonical amplitude

$$\alpha(t) = 2\sqrt{\epsilon_0\omega\mathcal{V}}\mathcal{A}(t) \tag{75}$$

- Canonical position and momentum:

$$\alpha(t) = x_c + ip_c \ , \tag{76}$$

i.e.

$$x_c = 2\sqrt{\epsilon_0\omega\mathcal{V}}x \qquad \text{and} \qquad p_c = 2\sqrt{\epsilon_0\omega\mathcal{V}}p \tag{77}$$

Field energy

Canonical variables

Mode energy

$$H = \frac{\omega}{2}\left[x_c^2 + p_c^2\right] \tag{78}$$

and obviously

$$\frac{dx_c}{dt} = \frac{\partial H}{\partial p_c} \qquad \text{and} \qquad \frac{dp_c}{dt} = -\frac{\partial H}{\partial x_c} \tag{79}$$

Proper canonical variables. Note that the x_c and p_c coordinates are not dimensionless (their joint dimension is the square root of an action)

Field momentum

Total momentum

Density of momentum proportional to the Poynting vector

$$\mathbf{g} = \frac{\mathbf{\Pi}}{c^2} \qquad \text{with} \qquad \mathbf{\Pi} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \tag{80}$$

The plane wave mode basis is most convenient to describe the momentum

$$\mathbf{E}^+(\mathbf{r}, t) = \sum_\ell \mathbf{E}_\ell^+ = \sum_\ell \; i\omega_\ell \mathcal{A}_\ell(t) \boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \tag{81}$$

and

$$\mathbf{B}^+(\mathbf{r}, t) = \sum_\ell \mathbf{B}_\ell^+ = \sum_\ell \; \mathcal{A}_\ell(t)(i\mathbf{k}_\ell \times \boldsymbol{\epsilon}_\ell) e^{i\mathbf{k}_\ell \cdot \mathbf{r}} \tag{82}$$

Field momentum

Total momentum

Using orthogonalities of modes and polarizations

$$\mathbf{P} = \sum_\ell \mathbf{P}_\ell \tag{83}$$

with

$$\mathbf{P}_\ell = \epsilon_0 \int (\mathbf{E}_\ell^+ + \mathbf{E}_\ell^-) \times (\mathbf{B}_\ell^+ + \mathbf{B}_\ell^-) \tag{84}$$

and after a painful calculation

$$\mathbf{P}_\ell = 2\epsilon_0 \mathcal{V} \omega_\ell |\mathcal{A}_\ell|^2 \boldsymbol{\epsilon}_\ell \times (\mathbf{k}_\ell \times \boldsymbol{\epsilon}_\ell) \tag{85}$$

or, finally

$$\mathbf{P} = \frac{1}{2} \sum_\ell |\alpha_\ell|^2 \mathbf{k}_\ell \tag{86}$$

with a clear interpretation.

Field momentum

Angular momentum

Angular momentum density $\mathbf{r} \times \mathbf{g}$ and hence

$$\mathbf{J} = \epsilon_0 \int \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \, d^3\mathbf{r} \tag{87}$$

A difficult calculation leads to

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \ , \tag{88}$$

where

$$\mathbf{S} = \epsilon_0 \int \mathbf{E} \times \mathbf{A} \, d^3\mathbf{r} \tag{89}$$

is the field's 'intrinsic angular momentum' and

$$\mathbf{L} = \epsilon_0 \int d^3\mathbf{r} \sum_j E_j (\mathbf{r} \cdot \nabla) A_j \ , \qquad j = (x, y, z) \tag{90}$$

is the field's 'orbital angular momentum'.

Field momentum

Spin angular momentum

Plane wave basis with circular polarizations

$$\mathbf{S} = i\epsilon_0\mathcal{V}\sum_n \omega_n \left[\mathcal{A}_{n+}\mathcal{A}_{n+}^*(\boldsymbol{\epsilon}_+ \times \boldsymbol{\epsilon}_+^*) + \mathcal{A}_{n-}\mathcal{A}_{n-}^*(\boldsymbol{\epsilon}_- \times \boldsymbol{\epsilon}_-^*) - \text{c.c.}\right] \quad (91)$$

Using $\boldsymbol{\epsilon}_+ \times \boldsymbol{\epsilon}_+^* = \boldsymbol{\epsilon}_+ \times \boldsymbol{\epsilon}_- = -i\mathbf{u_k}$ and $\boldsymbol{\epsilon}_- \times \boldsymbol{\epsilon}_-^* = i\mathbf{u_k}$

$$\mathbf{S} = \frac{1}{2}\sum_n \left[|\alpha_{n+}|^2 - |\alpha_{n-}|^2\right]\mathbf{u_k} \quad (92)$$

with an equally simple interpretation.

Field quantization

The field is a collection of independent harmonic oscillators. Let us quantify all of them independently, using the Dirac approach. The conjugate classical variables x_c and p_c should be replaced by two operators X and P (position and momentum operators, dimension also the square root of an action) acting in an infinite dimension Hilbert space, with the commutation rule:

$$[X, P] = i\hbar \tag{93}$$

Field quantization

Annihilation and creation operators

$$a = \frac{1}{\sqrt{2\hbar}}(X + iP) \tag{94}$$

and

$$a^\dagger = \frac{1}{\sqrt{2\hbar}}(X - iP) \tag{95}$$

with

$$\left[a, a^\dagger\right] = \mathbb{1} \tag{96}$$

Or

$$X = \sqrt{\frac{\hbar}{2}}\left(a + a^\dagger\right) \tag{97}$$

and

$$P = i\sqrt{\frac{\hbar}{2}}\left(a^\dagger - a\right) \tag{98}$$

Field quantization

Field quadratures

Define reduced units

$$X_0 = \frac{X}{\sqrt{2\hbar}} \qquad \text{and} \qquad P_0 = \frac{P}{\sqrt{2\hbar}} \tag{99}$$

With these definitions

$$[X_0, P_0] = \frac{i}{2} \tag{100}$$

$$a = X_0 + iP_0 \ , \qquad a^\dagger = X_0 - iP_0 \ , \qquad X_0 = \frac{a + a^\dagger}{2} \ , \qquad P_0 = i\frac{a^\dagger - a}{2} \tag{101}$$

Field quantization

Hamiltonian

$$H = \frac{\omega}{2}(X^2 + P^2) = \hbar\omega(X_0^2 + P_0^2) \tag{102}$$

or

$$H = \frac{\hbar\omega}{4}\left[(a + a^\dagger)^2 - (a^\dagger - a)^2\right] \tag{103}$$

and, in the 'normal order',

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{104}$$

whose diagonaization is described in all textbooks.

Field quantization

Number operator

$$N = a^\dagger a \tag{105}$$

Commutation relations:

$$[N, a] = -a \qquad \text{and} \qquad \left[N, a^\dagger\right] = a^\dagger \tag{106}$$

Eingenvalues: all positive integers, with nondegenerate eignestates

$$N\left|n\right\rangle = n\left|n\right\rangle \ , \tag{107}$$

Hence, the eigenergies are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{108}$$

Ground state: 'vacuum', $|0\rangle$, energy $\hbar\omega/2$

Field quantization

Fock states

$|n\rangle$ are the 'photon number states' with the orthogonality relation

$$\langle n\,|p\rangle = \delta_{n,p} \tag{109}$$

Annihilation and creation of photons with:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \tag{110}$$

with

$$a\,|0\rangle = 0 \tag{111}$$

and, similarly

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{112}$$

Hence

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle \tag{113}$$

Field quantization

All modes

$$H\left|n_1,\ldots,n_\ell\ldots\right\rangle = E_n\left|n_1,\ldots,n_\ell\ldots\right\rangle \tag{114}$$

with

$$E_n = \sum_\ell \left(n_\ell \hbar\omega_\ell + \frac{\hbar\omega_\ell}{2}\right) \tag{115}$$

and

$$\left|n_1,\ldots,n_\ell\ldots\right\rangle = \prod_\ell \frac{(a_\ell^\dagger)^{n_\ell}}{\sqrt{n_\ell!}}\left|0\right\rangle \tag{116}$$

Note that the vacuum state has an infinite energy (more on that later).

Field quantization

Vector potential operator

Classical normal variables:

$$\mathcal{A} = \frac{1}{2\sqrt{\epsilon_0 \omega \mathcal{V}}}(x_c + ip_c) \tag{117}$$

Corresponding quantum operators

$$A_\ell = \frac{1}{2\sqrt{\epsilon_0 \omega_\ell \mathcal{V}}}(X_\ell + iP_\ell) = \sqrt{\frac{\hbar}{2\epsilon_0 \omega_\ell \mathcal{V}}} a_\ell \tag{118}$$

Positive frequency vector potential

$$\mathbf{A}^+(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0 \omega_\ell \mathcal{V}}} a_\ell \mathbf{f}_\ell(\mathbf{r}) \tag{119}$$

Hermitian vector potential:

$$\mathbf{A}(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0 \omega_\ell \mathcal{V}}} \left(a_\ell \mathbf{f}_\ell(\mathbf{r}) + a_\ell^\dagger \mathbf{f}_\ell^*(\mathbf{r}) \right) \tag{120}$$

Field quantization

Electric field operator

The hermitian electric field is similarly:

$$\mathbf{E}(\mathbf{r}) = i\sum_{\ell} \mathcal{E}_\ell \left(a_\ell \mathbf{f}_\ell(\mathbf{r}) - a_\ell^\dagger \mathbf{f}_\ell^*(\mathbf{r}) \right) \tag{121}$$

where we define the 'field per photon in mode ℓ' by

$$\mathcal{E}_\ell = \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 \mathcal{V}}} \tag{122}$$

Field quantization

Magnetic field operator

$$\mathbf{B}(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}}\left(a_\ell\mathbf{h}_\ell(\mathbf{r}) + a_\ell^\dagger\mathbf{h}_\ell^*(\mathbf{r})\right) \tag{123}$$

with $\mathbf{h}_\ell = \boldsymbol{\nabla}\times\mathbf{f}_\ell$

Field quantization

Plane wave mode basis

$$\mathbf{A}^+(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}} a_\ell \boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{124}$$

$$\mathbf{E}^+(\mathbf{r}) = i\sum_\ell \mathcal{E}_\ell a_\ell \boldsymbol{\epsilon}_\ell e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{125}$$

$$\mathbf{B}^+(\mathbf{r}) = \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0\omega_\ell\mathcal{V}}} a_\ell (i\mathbf{k}_\ell \times \boldsymbol{\epsilon}_\ell) e^{i\mathbf{k}_\ell\cdot\mathbf{r}} \tag{126}$$

Field quantization

Heisenberg picture

Evolution of annihilation operator

$$i\hbar\frac{da_H}{dt} = [a_H, H] \qquad \text{i.e.} \qquad \frac{da_H}{dt} = -i\omega a_H \tag{127}$$

whose immediate solution is

$$a_H(t) = a_H(0)e^{-i\omega t} = ae^{-i\omega t} \tag{128}$$

Field quantization

Momentum, angular momentum

- Total momentum by replacing $|\alpha_\ell|^2$ in the classical expression by $\alpha_\ell^*\alpha_\ell$ and α_ℓ by $a_\ell\sqrt{2\hbar}$

$$\mathbf{P} = \sum_\ell \hbar\mathbf{k}_l\, a_\ell^\dagger a_\ell \qquad (129)$$

- Similarly

$$\mathbf{S} = \sum_n \hbar\mathbf{u}_{\mathbf{k}_n}[N_{n+} - N_{n-}] \qquad (130)$$

Field quantization

Field quadratures

Eigenstates of the quadratures:

$$X_0 |x\rangle = x |x\rangle \qquad \text{and} \qquad P_0 |p\rangle = p |p\rangle \tag{131}$$

Wavefunctions:

$$\Psi(x) = \langle x | \Psi\rangle \tag{132}$$

For the vacuum:

$$\Psi_0(x) = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2} \tag{133}$$

Also in the $|p\rangle$ representation:

$$\widetilde{\Psi}_0(p) = \left(\frac{2}{\pi}\right)^{1/4} e^{-p^2} \tag{134}$$

Suggests a pictorial representation of the vacuum as a small circle in phase plane.

Field quantization

Field quadratures

For the Fock state $|n\rangle$:

$$\Psi_n(x) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-x^2} H_n(x\sqrt{2}) \tag{135}$$

where H_n is the nth Hermite polynomial defined by

$$H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n} e^{-u^2} \tag{136}$$

These wavefunctions have n nodes and a a parity $(-1)^n$

Field quantization

Field quadratures

General field quadratures

$$X_\phi = \frac{ae^{-i\phi} + a^\dagger e^{i\phi}}{2} \tag{137}$$

Commutation:

$$\left[X_\phi, X_{\phi+\pi/2}\right] = \frac{i}{2} \tag{138}$$

Heisenberg relations

$$\Delta X_\phi \Delta X_{\phi+\pi/2} \geq \frac{1}{4} \tag{139}$$

Eigenstates $X_\phi \left|x_\phi\right\rangle = x_\phi \left|x_\phi\right\rangle$ with

$$\left|x_{\phi+\pi/2}\right\rangle = \frac{1}{\sqrt{\pi}} \int dy_\phi e^{2ix_{\phi+\pi/2}y_\phi} \left|y_\phi\right\rangle \tag{140}$$

Field quantization

Mode basis change

From basis $\mathbf{f}_\ell$ to $\mathbf{g}_p$, with

$$\mathbf{f}_\ell = \sum_p U_{\ell p}\mathbf{g}_p \tag{141}$$

where $U_{\ell p}$ is a unitary matrix that connects modes with identical frequencies.

The positive frequency part of the electric field can be written as:

$$\begin{aligned} \mathbf{E}^+ &= i\sum_\ell \mathcal{E}_\ell \mathbf{f}_\ell(\mathbf{r})a_\ell \\ &= i\sum_{\ell,p} \mathcal{E}_\ell U_{\ell p} a_\ell \mathbf{g}_p(\mathbf{r}) \\ &= \sum_p \mathcal{E}_p \mathbf{g}_p(\mathbf{r}) b_p \end{aligned} \tag{142}$$

Field quantization

Mode basis change

Defines the new annihilation operators

$$b_p = \sum_\ell U_{\ell p} a_\ell \tag{143}$$

and using unitarity $U^*_{\ell p} = U^\dagger_{p\ell}$

$$b_p^\dagger = \sum_\ell U^\dagger_{p\ell} a_\ell^\dagger \tag{144}$$

Field quantization

Mode basis change

Exercise: check new bosonic commutation rules

$$\begin{aligned}\left[b_p, b_q^\dagger\right] &= \sum_{\ell,m} U_{\ell p} a_\ell U_{qm}^\dagger a_m^\dagger - U_{qm}^\dagger a_m^\dagger U_{\ell p} a_\ell \\ &= \sum_{\ell,m} U_{\ell p} U_{qm}^\dagger \left[a_\ell, a_m^\dagger\right] \\ &= \sum_{\ell} U_{q\ell}^\dagger U_{\ell p} \\ &= \delta_{p,q} \qquad (145)\end{aligned}$$

Fock states

A basis of the Hilbert space

$$|\Psi\rangle = \sum_n c_n |n\rangle \tag{146}$$

Photon number distribution

$$p_n = |c_n|^2 \tag{147}$$

Mean number of photons

$$\overline{n} = \sum_n n p_n \tag{148}$$

Photon number variance

$$\begin{aligned} \Delta N^2 &= \langle N^2 \rangle - \langle N \rangle^2 \\ &= \sum_n (n - \overline{n})^2 p_n \end{aligned} \tag{149}$$

Fock states

Statistical mixtures

$$\rho = \sum_{n,p} \rho_{np} |n\rangle \langle p| \tag{150}$$

Photon number distribution

$$\rho_{nn} = p_n \tag{151}$$

Note that Fock states are not invariant in a mode basis change

$$|n_p\rangle = \frac{(b_p^\dagger)^{n_p}}{\sqrt{n!}} |0\rangle = \frac{\left(\sum_\ell U_{p\ell}^\dagger a_\ell^\dagger\right)^{n_p}}{\sqrt{n!}} |0\rangle \tag{152}$$

Fock states

Non classicality of Fock states

Fock states are very non-classical

- A large energy
- Zero average fields and potentials since $\langle n| a |n\rangle = 0$

Can we find more intuitive field states? Yes: Coherent states.

Coherent states

Displacement operator

A unitary defined by:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{153}$$

where α is an arbitrary complex amplitude

$$\alpha = \alpha' + i\alpha'' \tag{154}$$

$$D(\alpha)^\dagger D(\alpha) = \mathbb{1} \tag{155}$$

and

$$D(\alpha)^\dagger = D(-\alpha) \tag{156}$$

Coherent states

Displacement operator

An equivalent expression

$$D(\alpha) = e^{2i\alpha'' X_0 - 2i\alpha' P_0} \tag{157}$$

Using the Glauber relation

$$e^A e^B = e^{A+B} e^{[A,B]/2} \tag{158}$$

valid when

$$[A,[A,B]] = [B,[A,B]] = 0 \tag{159}$$

$$D(\alpha) = e^{-i\alpha'\alpha''} e^{2i\alpha'' X_0} e^{-2i\alpha' P_0} \tag{160}$$

a product of displacement operators:

$$e^{-2i\alpha' P_0} |x\rangle = |x + \alpha'\rangle \tag{161}$$

$$e^{2i\alpha'' X_0} |p\rangle = |p + \alpha''\rangle \tag{162}$$

Coherent states

Combination of displacements

Using Glauber

$$D(\alpha)D(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2} D(\alpha + \beta) \tag{163}$$

Note that

$$\Phi = (\alpha\beta^* - \alpha^*\beta)/2i = \frac{\alpha''\beta' - \alpha'\beta''}{2} \tag{164}$$

surface of the triangle with sides α and β.

Coherent states

Displacement of annihilation

Compute $D(-\alpha)aD(\alpha)$. Use Baker-Hausdorff lemma

$$e^A a e^{-A} = a + [A, a] + \frac{1}{2!}[A, [A, a]] + \dots \tag{165}$$

for $A = -\alpha a^\dagger + \alpha^* a$, with $[A, a] = \alpha$. Hence

$$D(-\alpha)aD(\alpha) = a + \alpha \mathbb{1} \tag{166}$$

Coherent states

Definition

The coherent states are defined as

$$|\alpha\rangle = D(\alpha)\,|0\rangle \ . \tag{167}$$

Note that $|0\rangle$ is a coherent state. Coherent states in general are the vacuum displaced by the complex amplitude α.
Wavefunction of a coherent state in the X_0 representation:

$$\Psi_\alpha(x) \propto e^{-(x-\alpha')^2} \tag{168}$$

and in the P_0 representation:

$$\widetilde{\Psi}_\alpha(p) \propto e^{-(p-\alpha'')^2} \tag{169}$$

Coherent states

Properties

- Right-eigenstates of the annihilation operator

$$a\left|\alpha\right\rangle = aD(\alpha)\left|0\right\rangle = D(\alpha)D(-\alpha)aD(\alpha)\left|0\right\rangle = (a+\alpha\mathbb{1})\left|0\right\rangle = \alpha\left|\alpha\right\rangle \tag{170}$$

since $a\left|0\right\rangle = 0$. Hence

$$\left\langle\alpha\right| a \left|\alpha\right\rangle = \alpha \qquad \text{and} \qquad \left\langle\alpha\right| a^\dagger \left|\alpha\right\rangle = \alpha^* \tag{171}$$

- Field operators have nonzero eigenvalues in the coherent states:

$$\langle \mathbf{E} \rangle = i\mathcal{E}\left(\mathbf{f}(\mathbf{r})\alpha - \mathbf{f}^*(\mathbf{r})\alpha^*\right) \tag{172}$$

$$\langle \mathbf{A} \rangle = \frac{\mathcal{E}}{\omega}\left(\mathbf{f}(\mathbf{r})\alpha + \mathbf{f}^*(\mathbf{r})\alpha^*\right) \tag{173}$$

Coherent states

Properties

- Average photon number

$$\overline{n} = \langle\alpha| a^\dagger a |\alpha\rangle = |\alpha|^2 \tag{174}$$

- Photon number variance. Using $N^2 = a^\dagger a a^\dagger a = (a^\dagger)^2 a^2 + a^\dagger a$

$$\left\langle N^2 \right\rangle = |\alpha|^4 + |\alpha|^2 \tag{175}$$

and

$$\Delta N^2 = |\alpha|^2 = \overline{n} \tag{176}$$

$$\frac{\Delta N}{\overline{n}} = \frac{1}{\sqrt{\overline{n}}} \tag{177}$$

Coherent states

Properties

- Expansion on the Fock state basis

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} \tag{178}$$

with $a|0\rangle = 0$:

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle \tag{179}$$

Expand exponential:

$$|\alpha\rangle = \sum_n c_n |n\rangle \ , \tag{180}$$

with

$$c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \tag{181}$$

Coherent states

Properties

- Photon number distribution

$$p_n = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = e^{-\overline{n}}\frac{\overline{n}^n}{n!} \tag{182}$$

For large average photon numbers

$$p_n \propto e^{-(n-\overline{n})^2/\overline{n}} \tag{183}$$

- Scalar product of coherent states

$$\begin{aligned} \langle\alpha\,|\beta\rangle &= e^{-(|\alpha|^2+|\beta|^2)/2}\sum_{n,p}\frac{(\alpha^*)^n\beta^p}{\sqrt{n!p!}}\langle n\,|p\rangle \\ &= e^{-(|\alpha|^2+|\beta|^2)/2}e^{\alpha^*\beta} \end{aligned} \tag{184}$$

Square modulus

$$|\langle\alpha\,|\beta\rangle\,|^2 = e^{-|\alpha-\beta|^2} \tag{185}$$

Coherent states

Properties

- Overcomplete basis

$$\mathbb{1} = \frac{1}{\pi}\int d^2\alpha \, |\alpha\rangle\langle\alpha| \tag{186}$$

Demonstration:

$$\int d^2\alpha \, |\alpha\rangle\langle\alpha| = \sum_{n,p} \frac{1}{\sqrt{n!p!}} |n\rangle\langle p| \int d^2\alpha \, e^{-|\alpha|^2} \alpha^n (\alpha^*)^p \tag{187}$$

Switch to polar coordinates $\alpha = \rho \exp(i\theta)$

$$\int \rho d\rho d\theta \, e^{-\rho^2} \rho^{n+p} e^{i\theta(n-p)} \tag{188}$$

Cancels when $n \neq p$.

Coherent states

Properties

- Overcomplete basis

  For $n = p$

  $$I_n = \pi \int du\, u^n e^{-u} \tag{189}$$

  with $u = \rho^2$. Integration per parts leads to $I_n = nI_{n-1}$ and $I_n = \pi n!$. Hence

  $$\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \pi \sum_n |n\rangle\langle n| \tag{190}$$

Coherent states

Properties

- Overcomplete basis
  Expansion is not uniquely defined:

$$|0\rangle = \frac{1}{\pi}\int d^2\alpha\, e^{-|\alpha|^2/2}\,|\alpha\rangle \tag{191}$$

and

$$|n\rangle = \frac{1}{\pi\sqrt{n!}}\int d^2\alpha\, e^{-|\alpha|^2/2}(\alpha^*)^n\,|\alpha\rangle \tag{192}$$

Coherent states

Properties

- Evolution

$$|\Psi(0)\rangle = |\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{193}$$

$$\begin{aligned} |\Psi(t)\rangle &= e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-in\omega t} e^{-i\omega t/2}|n\rangle \\ &= e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle \end{aligned} \tag{194}$$

Evolution of the amplitude is the same as in classical physics

$$\alpha(t) = \alpha(0)e^{-i\omega t} \tag{195}$$

Phase space representations

Seeks an analogue of the classical phase space distributions $f(x,p)$ of statistical physics allowing us to compute any average by

$$\overline{o} = \int f(x,p) o(x,p)\, dxdp \tag{196}$$

Transpose that to a field statistical mixture defined by the density operator ρ.

Phase space representations

Characteristic functions

Three operators ordering:

- Normal: a on right. e.g. number operator $a^\dagger a$
- Symmetric e.g. $(aa^\dagger + a^\dagger a)$
- Anti-Normal e.g. $aa^\dagger$

Any operator expression can be put in one of these forms by proper commutations of creation and annihilation operators.
Leads to three characteristic functions characterizing ρ

Phase space representations

Symmetric characteristic function

- Symmetric characteristic function

$$C_s^{[\rho]}(\lambda) = \langle D(\lambda) \rangle = \mathrm{Tr}\left[\rho e^{\lambda a^\dagger - \lambda^* a}\right] \tag{197}$$

with

$$C_s^{[\rho]}(0) = \mathrm{Tr}(\rho) = 1 \ . \tag{198}$$

D being unitary, all its eigenvalues have a unit modulus. Hence

$$|C_s^{[\rho]}(\lambda)| \leq 1 \tag{199}$$

and

$$C_s^{[\rho]}(-\lambda) = \left[C_s^{[\rho]}(\lambda)\right]^* \tag{200}$$

For a pure state

$$C_s^{[|\Psi\rangle\langle\Psi|]} = \langle\Psi| D(\lambda) |\Psi\rangle \tag{201}$$

Phase space representations

Normal and anti-normal characteristic functions

- Normal characteristic function

$$C_n^{[\rho]}(\lambda) = \mathrm{Tr}\left[\rho e^{\lambda a^\dagger} e^{-\lambda^* a}\right] \tag{202}$$

- Anti-normal characteristic function

$$C_{an}^{[\rho]}(\lambda) = \mathrm{Tr}\left[\rho e^{-\lambda^* a} e^{\lambda a^\dagger}\right] \tag{203}$$

- Relations

$$C_n^{[\rho]}(\lambda) = e^{|\lambda|^2/2} C_s^{[\rho]}(\lambda) \quad C_{an}^{[\rho]}(\lambda) = e^{-|\lambda|^2/2} C_s^{[\rho]}(\lambda) \tag{204}$$

Phase space representations

The Husimi-Q representation

Definition:

$$Q^{[\rho]}(\alpha) = \frac{1}{\pi^2} \int d^2\lambda\, e^{(\alpha\lambda^* - \alpha^*\lambda)} C_{an}^{[\rho]}(\lambda) \tag{205}$$

After some algebra:

$$Q^{[\rho]}(\alpha) = \frac{1}{\pi}\mathrm{Tr}\left[\rho \left|\alpha\right\rangle \left\langle\alpha\right|\right] = \frac{1}{\pi}\left\langle\alpha\right| \rho \left|\alpha\right\rangle = \frac{1}{\pi}\mathrm{Tr}[\left|0\right\rangle \left\langle 0\right| D(-\alpha)\rho D(\alpha)] \tag{206}$$

The Q distribution is positive, bounded by $1/\pi$ and normalized ($\int d^2\alpha\, Q(\alpha) = 1$).

Phase space representations

The Husimi-Q representation

A few states

- Coherent state $|\beta\rangle$

$$Q^{[|\beta\rangle\langle\beta|]}(\alpha) = \frac{1}{\pi}|\langle\alpha|\beta\rangle|^2 = \frac{1}{\pi}e^{-|\alpha-\beta|^2} \tag{207}$$

- Fock state $|n\rangle$

$$Q^{[|n\rangle\langle n|]}(\alpha) = \frac{1}{\pi}\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2} \tag{208}$$

Phase space representations

The Husimi-Q representation

- Cat state

$$\left|\Psi_{\text{cat}}^{\pm}\right\rangle = \frac{1}{\sqrt{\mathcal{N}_{\pm}}}\left(|\beta\rangle \pm |-\beta\rangle\right) \tag{209}$$

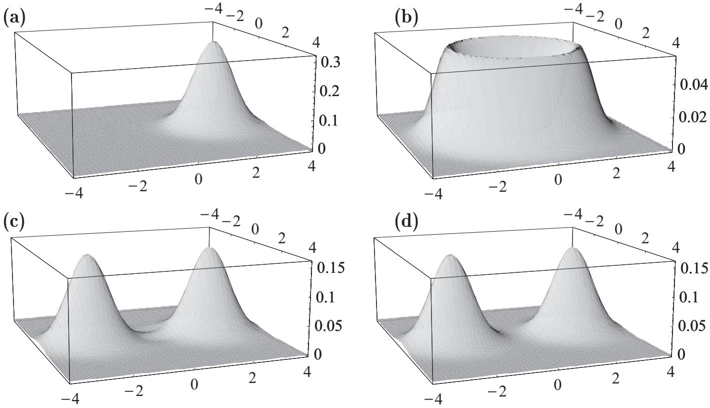
where:

$$\mathcal{N}_{\pm} = 2\left(1 \pm e^{-2|\beta|^2}\right) \tag{210}$$

$$Q^{[\text{cat},\pm]}(\alpha) = \frac{1}{\pi\mathcal{N}_{\pm}}\left[e^{-|\alpha-\beta|^2} + e^{-|\alpha+\beta|^2} \pm 2e^{-(|\alpha|^2+|\beta|^2)}\cos(2\beta\alpha'')\right] \tag{211}$$

Phase space representations

The Husimi-Q representation



(a) Coherent state $|\beta\rangle$, with $\beta = \sqrt{5}$. (b) Five-photon Fock state. (c) Schrödinger cat state, superposition of two coherent fields $|\pm\beta\rangle$, with $\beta = \sqrt{5}$. (d) Statistical mixture of the same coherent components.

Phase space representations

The Wigner function

Definition:

$$W(\alpha) = \frac{1}{\pi^2} \int d^2\lambda \, C_s(\lambda) e^{\alpha\lambda^* - \alpha^*\lambda} \tag{212}$$

After a long derivation (see complete lecture notes)

$$W(x,p) = \frac{2}{\pi} \mathrm{Tr}[D(-\alpha)\rho D(\alpha)\mathcal{P}] \tag{213}$$

where the unitary parity operator $\mathcal{P}$ is defined by

$$\mathcal{P}\,|x\rangle = |-x\rangle \;\; ; \quad \mathcal{P}\,|p\rangle = |-p\rangle \tag{214}$$

Phase space representations

The Wigner function

Properties of parity operator

$$\mathcal{P}\,|n\rangle = (-1)^n\,|n\rangle \tag{215}$$

and hence

$$\mathcal{P} = e^{i\pi a^\dagger a} \tag{216}$$

The modulus of its average is lower than one. Thus

$$-2/\pi \leq W(\alpha) \leq 2/\pi \tag{217}$$

Phase space representations

The Wigner function

Marginals of the Wigner distribution:

$$P(x) = \langle x|\,\rho\,|x\rangle = \int dp\, W(x,p) \tag{218}$$

and

$$P(p) = \langle p|\,\rho\,|p\rangle = \int dx\, W(x,p) \tag{219}$$

More generally,

$$P(p_\phi) = \int dx_\phi\, W(x_\phi, p_\phi) \tag{220}$$

with

$$x_\phi = x\cos\phi + p\sin\phi\ ; \quad p_\phi = -x\sin\phi + p\cos\phi \tag{221}$$

Phase space representations

The Wigner function

The average of any operator can be directly obtained from the Wigner function

$$\langle O \rangle = \int dxdp\, W(x,p) o_s(x,p) \qquad (222)$$

where o_s is the symmetrized form of the operator O in terms of the field quadratures.

Phase space representations

The Wigner function

A few states

- Coherent state

$$W^{[|\beta\rangle\langle\beta|]}(\alpha) = \frac{2}{\pi} e^{-2|\beta-\alpha|^2} \tag{223}$$

- Thermal field

$$W^{[\rho_{\rm th}]}(\alpha) = \frac{2}{\pi}\frac{1}{2n_{\rm th}+1} e^{-2|\alpha|^2/(2n_{\rm th}+1)} \tag{224}$$

Phase space representations

The Wigner function

- Squeezed vacuum $S(\xi)\left|0\right\rangle$ with

$$S(\xi) = e^{(\xi^* a^2 - \xi a^{\dagger 2})/2} \tag{225}$$

Reduced fluctuations on X_0

$$\Delta X_0 = \frac{1}{2} e^{-\xi} \tag{226}$$

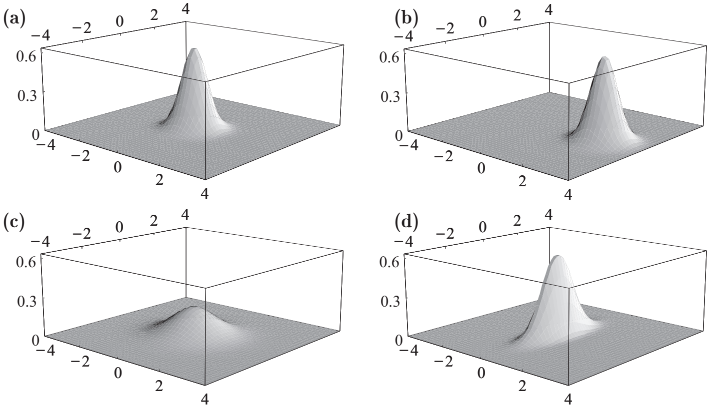
and

$$\Delta P_0 = \frac{1}{2} e^{\xi} \tag{227}$$

$$W^{[sq,\xi]}(x,p) = \frac{2}{\pi} e^{-2\exp(2\xi)x^2} e^{-2\exp(-2\xi)p^2} \tag{228}$$

Phase space representations

The Wigner function



(a) Vacuum state. (b) Coherent state with $\beta = \sqrt{5}$. (c) Thermal field with $n_{\text{th}} = 1$ photon on the average. (d) A squeezed vacuum state, with a squeezing parameter $\xi = 0.5$.

Phase space representations

The Wigner function

- Fock state

$$W^{[|n\rangle\langle n|]}(\alpha) = \frac{2}{\pi}(-1)^n e^{-2|\alpha|^2}\mathcal{L}_n(4|\alpha|^2) \tag{229}$$

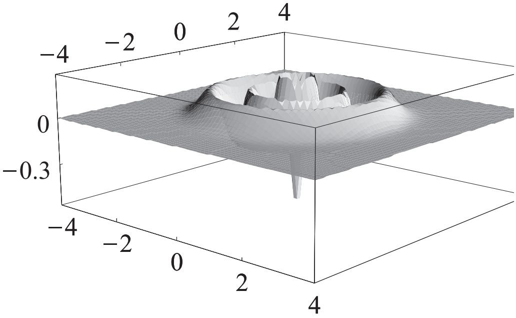
with

$$W^{[|n\rangle\langle n|]}(0) = \frac{2}{\pi}(-1)^n \tag{230}$$

$$W^{[|1\rangle\langle 1|]}(\alpha) = -\frac{2}{\pi}(1-4|\alpha|^2)e^{-2|\alpha|^2} \tag{231}$$

Phase space representations

The Wigner function



Wigner function of a five-photon Fock state.

Phase space representations

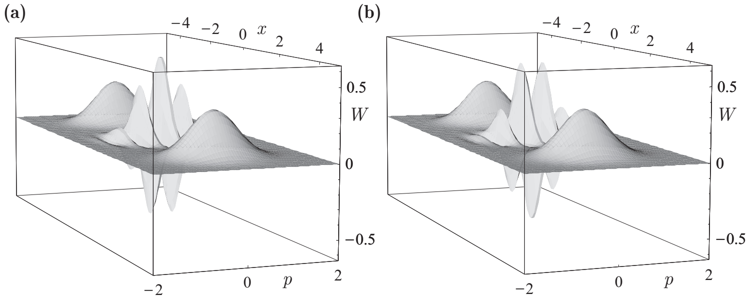
The Wigner function

- Cat state

$$W^{[\mathrm{cat},\pm]}(\alpha) = \frac{1}{\pi(1 \pm e^{-2|\beta|^2})}\left[e^{-2|\alpha-\beta|^2} + e^{-2|\alpha+\beta|^2} \pm 2e^{-2|\alpha|^2}\cos(4\alpha''\beta)\right] \tag{232}$$

Phase space representations

The Wigner function

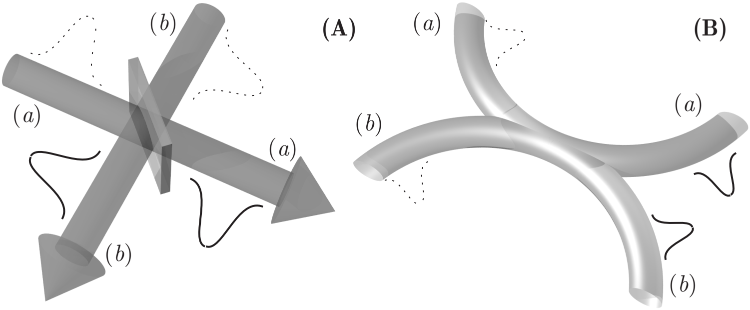


Wigner functions of even (a) and odd (b) 10-photon π-phase cats. The Wigner function provides a clear depiction of the non-classical features of a quantum state.

Beamsplitter

Coupling field modes

A simple model for coupling two modes of the radiation field



Beamsplitter

Classical model

Transformation of the electric field amplitudes

$$\begin{pmatrix} E'_a \\ E'_b \end{pmatrix} = U_c \begin{pmatrix} E_a \\ E_b \end{pmatrix} = \begin{pmatrix} t(\omega) & r(\omega) \\ r(\omega) & t(\omega) \end{pmatrix} \begin{pmatrix} E_a \\ E_b \end{pmatrix} \tag{233}$$

where the unitary U_c can also be written in a simple case as

$$U_c(\theta) = \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{234}$$

Quantum beamsplitter

Hamiltonian model

Model the beamsplitter action as a transient application of the Hamiltonian

$$H_{ab}(t) = -\hbar \frac{g(t)}{2}(ab^\dagger + a^\dagger b) \tag{235}$$

a and b: annihilation operators; $g(t)$ slowly varying real function

Quantum beamsplitter

Heisenberg point of view

Transformation of the annihilation operator:

$$a' = U^\dagger a U \tag{236}$$

where

$$U = e^{-(i/\hbar)\int H_{ab}(t)\,dt} = e^{-iG\theta/2} \tag{237}$$

with

$$G = -(ab^\dagger + a^\dagger b) \quad \text{and} \quad \theta = \int g(t)\,dt \tag{238}$$

Using Baker-Hausdorff

$$\begin{aligned} a' &= U^\dagger a U = e^{iG\theta/2} a e^{-iG\theta/2} = a + \frac{i\theta}{2}[G,a] \\ &+ \frac{i^2\theta^2}{2!2^2}[G,[G,a]] + \cdots + \frac{i^n\theta^n}{n!2^n}[G,[G,[\cdots,[G,a]]]] + \cdots \end{aligned} \tag{239}$$

Quantum beamsplitter

Heisenberg point of view

With $[G, a] = b$ and $[G, [G, a]] = a$, series sum up to

$$a' = U^\dagger a U = \cos(\theta/2)\, a + i \sin(\theta/2)\, b \qquad (240)$$

and similarly:

$$b' = U^\dagger b U = i \sin(\theta/2)\, a + \cos(\theta/2)\, b \qquad (241)$$

Noting that $U^\dagger(\theta) = U(-\theta)$

$$U a^\dagger U^\dagger = \cos(\theta/2)\, a^\dagger + i \sin(\theta/2)\, b^\dagger \; ; \qquad U b^\dagger U^\dagger = i \sin(\theta/2)\, a^\dagger + \cos(\theta/2)\, b^\dagger \qquad (242)$$

Quantum beamsplitter

State transformations

Transformation of some simple states:

- No photon: $|\Psi\rangle = |0,0\rangle$. This state is obviously invariant
- One photon in mode a

$$U|1,0\rangle = Ua^\dagger|0,0\rangle = Ua^\dagger U^\dagger U|0,0\rangle = Ua^\dagger U^\dagger|0,0\rangle \tag{243}$$

and, using the Heisenberg point of view results in:

$$\begin{aligned} U|1,0\rangle &= \left[\cos(\theta/2)\,a^\dagger + i\sin(\theta/2)\,b^\dagger\right]|0,0\rangle \\ &= \cos(\theta/2)\,|1,0\rangle + i\sin(\theta/2)\,|0,1\rangle \end{aligned} \tag{244}$$

- One photon in mode b

$$\begin{aligned} U|0,1\rangle &= \left[i\sin(\theta/2)\,a^\dagger + \cos(\theta/2)\,b^\dagger\right]|0,0\rangle \\ &= i\sin(\theta/2)\,|1,0\rangle + \cos(\theta/2)\,|0,1\rangle \end{aligned} \tag{245}$$

Quantum beamsplitter

State transformations

- n photons

$$U|n,0\rangle = U\frac{(a^\dagger)^n}{\sqrt{n!}}|0,0\rangle = \frac{1}{\sqrt{n!}}U(a^\dagger)^n U^\dagger U|0,0\rangle \qquad (246)$$

With $U(a^\dagger)^n U^\dagger = (Ua^\dagger U^\dagger)^n$,

$$U|n,0\rangle = \frac{1}{\sqrt{n!}}\left[\cos\frac{\theta}{2}\,a^\dagger + i\sin\frac{\theta}{2}\,b^\dagger\right]^n|0,0\rangle \qquad (247)$$

expansion of the r.h.s.

$$U|n,0\rangle = \sum_{p=0}^{n}\binom{n}{p}^{1/2}[\cos(\theta/2)]^{n-p}\,[i\sin(\theta/2)]^p\,|n-p,p\rangle \qquad (248)$$

Quantum beamsplitter

State transformations

- n photons, balanced splitter $(\theta = \pi/2)$

$$U(\pi/2, 0)\,|n, 0\rangle = \frac{1}{\sqrt{2^n}} \sum_{p=0}^{n} \binom{n}{p}^{1/2} (i)^p\,|n-p, p\rangle \tag{249}$$

  - Random output selection for each photon
  - A massively entangled state of the two output modes

Quantum beamsplitter

State transformations

- Coherent state $|\alpha\rangle$

$$U|\alpha,0\rangle = UD_a(\alpha)U^\dagger|0,0\rangle \quad (250)$$

rewrites, with $Uf(A)U^\dagger = f(UAU^\dagger)$

$$UD(\alpha)U^\dagger = e^{\alpha Ua^\dagger U^\dagger - \alpha^* UaU^\dagger} \quad (251)$$

and

$$U|\alpha,0\rangle = D_a[\alpha\cos(\theta/2)]\,D_b[i\alpha\sin(\theta/2)]\,|0,0\rangle \quad (252)$$

finally

$$U|\alpha,0\rangle = |\alpha\cos(\theta/2), i\alpha\sin(\theta/2)\rangle \quad (253)$$

An unentangled states, with two coherent amplitudes split according to the classical laws.

Quantum beamsplitter

State transformations

- Photon collision on a beamsplitter

$$U|1,1\rangle = Ua^\dagger b^\dagger |0,0\rangle = Ua^\dagger U^\dagger U b^\dagger U^\dagger |0,0\rangle \tag{254}$$

Hence:

$$U|1,1\rangle = \frac{i\sin\theta}{\sqrt{2}}\left[|2,0\rangle + |0,2\rangle\right] + \cos\theta\,|1,1\rangle \tag{255}$$

which is, in general, an entangled state. Balanced beam-splitter ($\theta = \pi/2$):

$$U(\pi/2,0)\,|1,1\rangle = \left(|2,0\rangle + |0,2\rangle\right)/\sqrt{2} \tag{256}$$

Photon bunching due to their bosonic nature.

Relaxation

Jump operators

Learn how to treat the coupling of a field mode to the external world.

Examples of physical situations

- Propagation of a beam in a diffusive medium
- Field in a cavity with output coupling (laser)
- Field in a box with imperfect conductivity (real cavity)

Relaxation

Jump operators

Only two possible jump operators at finite temperature T

- $L_- = \sqrt{\kappa_-} a$: loss of a photon in the environment (even when $T = 0$)
- $L + - = \sqrt{\kappa_+} a^\dagger$: creation of a thermal excitation

Jump rates linked to the temperature of the environment

$$\kappa_+ = \kappa_- e^{-\hbar\omega/k_b T} \tag{257}$$

Using

$$n_{\text{th}} = \frac{1}{e^{\hbar\omega/k_b T} - 1} \tag{258}$$

we get

$$\frac{\kappa_-}{\kappa_+} = \frac{1 + n_{\text{th}}}{n_{\text{th}}} \tag{259}$$

and write

$$\kappa_- = \kappa(1 + n_{\text{th}}) \; ; \qquad \kappa_+ = \kappa n_{\text{th}} \tag{260}$$

Relaxation

Lindblad equation

$$\frac{d\rho}{dt} = -i\omega_c \left[a^\dagger a, \rho\right] - \frac{\kappa(1+n_{\rm th})}{2}\left(a^\dagger a\rho + \rho a^\dagger a - 2a\rho a^\dagger\right) - \frac{\kappa n_{\rm th}}{2}\left(aa^\dagger\rho + \rho aa^\dagger - 2a^\dagger\rho a\right) \tag{261}$$

where we have discarded the vacuum energy. Note that all of the Hamiltonian part can be removed by an interaction representation (relaxation terms unchanged). For the photon number distribution:

$$\frac{dp(n)}{dt} = \kappa(1+n_{\rm th})(n+1)p(n+1) + \kappa n_{\rm th} np(n-1) - [\kappa(1+n_{\rm th})n + \kappa n_{\rm th}(n+1)]p(n) \tag{262}$$

Relaxation

Thermal equilibrium

Detailed balance argument

$$\kappa(1 + n_{\rm th})np(n) = \kappa n_{\rm th} np(n-1) \tag{263}$$

leading to:

$$\frac{p(n)}{p(n-1)} = \frac{n_{\rm th}}{1 + n_{\rm th}} = e^{-\hbar\omega/k_b T} \tag{264}$$

The expected Maxwell equilibrium

Relaxation

Fock states

At $T = 0$, relaxation of a Fock state

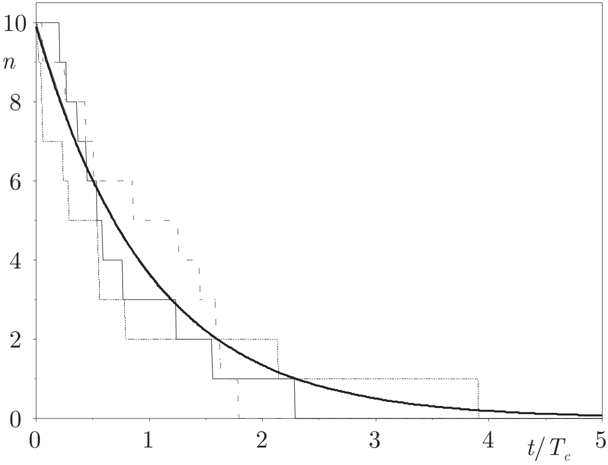
- Jump : removal of a photon
- No jump: non hermitian Hamiltonian

$$H_e = -i\hbar J = -i\hbar\kappa a^\dagger a/2 \tag{265}$$

Leaves photon number states invariant

Relaxation

Fock states



Relaxation of a 10-photon Fock state.

Relaxation

Coherent state

Monte Carlo trajectory

- Jump: no evolution since $|\alpha\rangle$ is an eingenstate of a
- No jumps: evolution with non hermitian hamiltonian, equivalent to a complex mode frequency

$$|\beta\rangle \rightarrow \left|\beta e^{-\kappa\tau/2}\right\rangle \tag{266}$$

A coherent state remains coherent, with an exponentially damped amplitude.

Relaxation

Coherent state

No change of the photon number in a quantum jump ? A bayesian argument. $p(n|c)$ photon number distribution before the jump knowing that a jump occurs ('click' in the environment.) With

$$p(n,c) = p(c|n)p(n) = p(n|c)p_c \tag{267}$$

$$p(n|c) = p(n)\frac{p(c|n)}{p_c} = \frac{n}{\overline{n}}p(n) = e^{-\overline{n}}\frac{\overline{n}^{n-1}}{(n-1)!} = p(n-1) \tag{268}$$

A translated Poisson distribution with $\overline{n}+1$ photons on the average. After jump photon number unchanged. Explains why the photon number distribution is invariant in a jump. Specific property of coherent states.