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Internship Report

by

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It has been around half year that I stay in Quantum Optics Group, Laboratoire Kastler Brossel, UPMC. From one day per week as a course called Research Project to full-time as an internship, I learned not only the fundamental knowledge of quantum optics but new theory and experiment in this field. Staying here also gives me an opportunity to enjoy the beauty of Paris. From RER B Gare d'Orsay-Ville to Métro Jussieu, it is rather far but exciting, eventually I got used to this journey meanwhile I began missing it when it is about to end. The very beginning mind itself is the most accomplished mind of true enlightenment. The reason why I came to France is to chase my dream, now I am writing these words to thank the people who helped me on this way.

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Contents

Introduction

Light interact either like a particle or wave, when the wavelength of light is comparable to the inter atomic distance it behaves like a wave otherwise like a particle, which is called the wave-particle duality. Over the past century, the physicists such as, Max Planck, Einstein, Louis de Broglie, Arthur Compton, Niels Bohr and many others, make a contribution to this theory. A theory called quantum electrodynamics, established by RP Feynman, (QED) based mainly on classical electrodynamics, special relativity and quantum mechanics has been established to describe how light and matter interact. At that time the tools for studying light is rather mature, however the non-linear optics (NLO), a branch of optics, describing the behaviour of light in non-linear media was not noticed until the discovery of second-harmonic generation was found by by Peter Franken et al. at University of Michigan in 1961[1]. Up to now, the variety of phenomena not only second-harmonic generation but third-harmonic generation, sum-frequency generation, difference-frequency generation, as well as optical parametric oscillation [2, 3]aroused physicists' attention.

Let us consider the electric field that is due to the light itself, which means that the refractive index of the non-linear optic material changes in response to an electric field. This kind of media is called the Kerr media[4]. One of the application for Kerr media is to generate the squeezed light, which is initially observed in 1985[5], furthermore, the non-Linear process are a key resource in quantum optics to generate such non-classical state of light. However, the mathematical complexity inherent to non-linear processes requires to either performs numerical simulation, providing low physical insight, either propose some approximation. Often it is chosen to resolve the mean field and to study linearised dynamics of fluctuations which may reveal the quantum nature of the produced light. However, linearising the fluctuation dynamics discards many features and we propose to explore in some extended the role play by these too often neglected features going beyond linearisation of the fluctuations.

Chapter 1

An introduction to open quantum system

This chapter mainly introduces the basic knowledge of quantum mechanics. Open quantum system consisting of system itself and the environment plays a very important role in variety of quantum systems. There is no such a quantum system that is totally isolated from its surroundings in the real word, furthermore, we focus more on the interaction between the system and the environment instead of the closed quantum system consisting of the quantum system itself.

Figure 1.1: The quantum system is considered to be surrounded by the infinity large blank space representing the environment.

A system is considered as open as soon as it exchange energy or particles or information with an environment as shown in the Figure 1.1. An environment is an other system, larger, considered as not affected by the change induced by the exchange with the system of interests. By definitive the Universe is a close system as energy, particles and information is conserved. This chapter contains five sections, we mainly focus on the open quantum system and the interaction between the environment and the system. We will see later that these three parts containing the intrinsic property of the quantum mechanics are rather fundamental but essential as well.

1.1 Density operator and its time evolution

Under Schrödinger picture, the state vectors evolve with time but the operators are constant with respect to the time resulting in the Hamiltonian $H(t)$ evolving with time and the state $|\psi(t)\rangle$ satisfying the Schrödinger equation and its formal solution

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle \tag{1.1.1}$$

$$|\psi(t)\rangle = U(t;t_0)|\psi(t_0)\rangle \tag{1.1.2}$$

for simplicity, here $\hbar = 1$ is considered. More explicitly, the exponential solution in terms of the initial time t_0 and any time t has the form

$$U(t;t_0) = e^{-iH(t-t_0)} \tag{1.1.3}$$

The density operator which is self-adjoint, positive semi-definite, of trace one, in the case of any quantum system is given by

$$\rho(t) = |\psi(t)\rangle\langle\psi(t)| \tag{1.1.4}$$

Expression 1.1.4 can be expressed by any state of the radiation field as a diagonal sum over coherent states even though such states are nonorthogonal with

$$\rho(t) = \sum_n P_n|\psi_n(t)\rangle\langle\psi_n(t)| \tag{1.1.5}$$

where the $\psi_n(t)$ is the mixed state. t_0 refers to the initial time

$$\rho(t_0) = |\psi(t_0)\rangle\langle\psi(t_0)| \tag{1.1.6}$$

Using Equation (1.1.3) in Equation (1.1.5) and noting that for the "Bra part" appearing in Equation (1.1.5) we need to add the dagger of time revolution operator U

$$\rho(t) = \sum_n P_n U(t;t_0)|\psi_n(t_0)\rangle\langle\psi_n(t_0)|U^\dagger(t;t_0) \tag{1.1.7}$$

Checking the Equation (1.1.7) carefully and recalling Equation (1.1.6) yields

$$\rho(t) = U(t;t_0)\rho(t_0)U^\dagger(t;t_0) \tag{1.1.8}$$

Differentiating Equation (1.1.8), then inserting Equation (1.1.3) into the time derivative form gives the density operator evolution equation

$$\begin{aligned}\frac{\partial\rho(t)}{\partial t} &= -iHe^{-iH(t-t_0)}|\psi(t_0)\rangle\langle\psi(t_0)|e^{iH(t-t_0)} + ie^{-iH(t-t_0)}|\psi(t_0)\rangle\langle\psi(t_0)|e^{iH(t-t_0)}H \\ &= -i[H\rho(t) - \rho(t)H] \\ &= -i[H(t), \rho(t)]\end{aligned} \tag{1.1.9}$$

which is known as the Liouville-von Neumann equation. In such a closed system, the density operator has a series of properties that can be easily proved. It is Hermitian namely $\rho = \rho^\dagger$, and a pure state to take $\rho^2 = \rho$. The trace of this operator is always 1 such that $Tr\rho = 1$. These properties shall be used in the following chapters.

1.2 Interaction between the system and the environment

In real world quantum system are not isolated but interact with an external quantum system called the environment, bath or reservoir. One important character of the environment is that it is un-affected by the system especially when the latter changes. Here we are more interested in this interaction process instead of the system alone with splitting the whole Hamiltonian as two parts

$$H = H_0 + H_I \tag{1.2.1}$$

where H_0, consisting of H_S and H_B is respective the Hamiltonian of the system and the environment, is the non-interaction part; H_I represents the interaction part and is taken to be time-independent for simplicity in the formalism. Considering an arbitrary observable O, which is no intrinsic time dependence neither and the total density operator $\rho(t)$ and using Equation (1.1.8), the expected value of this observable shows

$$\begin{aligned}\langle O(t)\rangle &= Tr(O\rho(t)) \\ &= Tr(OU(t;t_0)\rho(t_0)U^\dagger(t;t_0))\end{aligned} \tag{1.2.2}$$

Similarly to how the Hamiltonian is treated at the beginning of this section, the time evolution can be separated into two parts corresponding to the propagator of H_0 and H_I

$$U(t;t_0) = U_0(t;t_0)U_I(t;t_0) \tag{1.2.3}$$

To get the more compact formula, taking Equation (1.2.3) into account and using the cyclic property of the trace $Tr(ABC) = Tr(CAB) = Tr(BCA)$ yields

$$\begin{aligned} \langle O(t)\rangle &= Tr(OU_0(t,t_0)U_I(t,t_0)\rho(t_0)U_I^\dagger(t,t_0)U_0^\dagger(t,t_0)) \\ &= Tr(U_0^\dagger(t,t_0)OU_0(t,t_0)U_I(t,t_0)\rho(t_0)U_I^\dagger(t,t_0)) \end{aligned} \tag{1.2.4}$$

The observable and density operator related to the interaction part can be esaily extracted

$$O_I = U_0^\dagger(t,t_0)OU_0(t,t_0) \tag{1.2.5}$$

$$\rho_I(t) = U_I(t,t_0)\rho(t_0)U_I^\dagger(t,t_0) \tag{1.2.6}$$

which imply that such operator corresponds to the interaction part of the density operator

$$\langle O(t)\rangle = Tr(O_I\rho_I(t)) \tag{1.2.7}$$

Taking into account Equation (1.2.3) and Equation (1.1.8), the interaction density operator is therefore

$$\begin{aligned} \rho_I(t) &= U_0^\dagger(t,t_0)U(t,t_0)\rho(t_0)U^\dagger(t,t_0)U_0(t,t_0) \\ &= U_0^\dagger(t,t_0)\rho(t)U_0(t,t_0) \end{aligned} \tag{1.2.8}$$

Recalling Equation (1.1.3) but changing from H to H_0, Equation (1.2.5) can be rewritten as

$$O_I = e^{iH_0(t-t_0)}Oe^{-iH_0(t-t_0)} \tag{1.2.9}$$

Time derivative for Equation (1.2.6) reads

$$\begin{aligned} \frac{\partial\rho_I(t)}{\partial t} &= \frac{\partial U_I(t,t_0)}{\partial t}\rho(t_0)U_I^\dagger(t,t_0) + U_I\frac{\partial\rho(t_0)}{\partial t}U_I^\dagger(t,t_0) + U_I(t,t_0)\rho(t_0)\frac{\partial U_I^\dagger(t,t_0)}{\partial t} \\ &= -iH_IU_I(t,t_0)\rho(t_0)U_I^\dagger(t,t_0) + U_I(t,t_0)\rho(t_0)U_I^\dagger(t,t_0)iH_I \\ &= -iH_I\rho_I(t) + iH_I\rho_I(t) \\ &= -i[H_I,\rho_I] \end{aligned} \tag{1.2.10}$$

with

$$H_I(t) = U_0^\dagger(t,t_0)H_IU_0(t,t_0) \tag{1.2.11}$$

Comparing this result with Equation (1.1.9), the above result reveals that the more general form is also correct when we consider only one part of the system. Further more we can directly put down the Equation (1.2.11) since system is combined by the tensor product of each part, however the action tensor product cannot change the intrinsic property of the density operator, in other words, for any components of the system, its density time evolution equation can be expressed in such a form[6], which is also what the iouville-von Neumann equation tells us.

1.3 Reduced density

The density operator $\rho(t)$ is considered to be the tensor product between system and environment

$$\rho(t) = \rho_S(t) \otimes \rho_B(t) + \rho_C(t) \tag{1.3.1}$$

where $\rho_S(t)$ and $\rho_B(t)$ is denoted the density of the system and the environment, respectively, $\rho_C(t)$ represents the high order in $H(t)$, but here we neglect with high order term by taking $\rho_C(t) = 0$. Considering the dimension of environment, for any observable of system it can always be written

$$O(t) = O_S(t) \otimes \mathbb{1}_B \tag{1.3.2}$$

If we want to calculate the mean value of this operator $O(t)$, the trace should be the combination of the system and the environment

$$\begin{aligned} \langle O(t) \rangle &= Tr_S Tr_B \left[O_S(t) \otimes \mathbb{1}_B \, \rho_S(t_0) \otimes \rho_B(t_0) \right] \\ &= Tr_S Tr_B \left[U(t; t_0) O(t_0) U^{\dagger}(t; t_0) \, \rho_S(t_0) \otimes \rho_B(t_0) \right] \\ &= Tr_S Tr_B \left[O(t_0) U^{\dagger}(t; t_0) \left(\rho_S(t_0) \otimes \rho_B(t_0) \right) U(t; t_0) \right] \\ &= Tr_S \left[O(t_0) \, \rho_S(t) \right] \end{aligned} \tag{1.3.3}$$

where $\rho_S(t)$ is the time-dependent reduced density matrix given by

$$\rho_S(t) = Tr_B \left[U^{\dagger}(t; t_0) \left(\rho_S(t_0) \otimes \rho_B(t_0) \right) U(t; t_0) \right] \tag{1.3.4}$$

which proves that the reduced operator completely describes all accessible information about system S, while having dimension much smaller than that of the combined system-environment density operator $\rho(t)$.

1.4 Born and Markov approximation

Following the former section under interaction picture (Dirac picture), we begin studying the Hamiltonian of the interaction term with respect to the time

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t} \tag{1.4.1}$$

Note that $H_I(t)$ is under the interaction picture. Directly take $t_0 = 0$. In the closed quantum system, there exists a corresponding mater equation. We assume that the combined system and environment is a closed quantum system. Then we obtain

$$\frac{\partial \rho_I(t)}{\partial t} = -i[H_I(t), \rho_I(t)] \tag{1.4.2}$$

We can clearly get the solution of this equation

$$\rho_I(t) = \rho(0) - i\int_0^t dt' [H_I(t'), \rho(t')] \tag{1.4.3}$$

Then we substitute back into (1.4.2) to give

$$\frac{\partial \rho(t)}{\partial t} = -i[H_I(t), \rho(0)] - \int_0^t ds [H_I(t), [H_I(t'), \rho(t')]] \tag{1.4.4}$$

Taking Equation (1.3.4) the trace over the environment gives the density of the system

$$\begin{aligned} \frac{\partial \rho_S(t)}{\partial t} &= -iTr_B[H_I(t), \rho(0)] - \int_0^t ds Tr_B[H_I(t), [H_I(t'), \rho(t')]] \\ &= -\int_0^t ds Tr_B[H_I(t), [H_I(t'), \rho(t')]] \end{aligned} \tag{1.4.5}$$

where the first term on the right-hand part is zero $-iTr_B[H_I(t), \rho(0)] = 0$ resulting from the The Born approximation is showing by the following definition

$$\rho(t) \approx \rho_S(t) \otimes \rho_B^{st} \tag{1.4.6}$$

where the ρ_B^{st} means, as discussed in beginning, the environment can be considered as a steady system. Substituting back into (1.4.6)

$$\frac{\partial \rho_S(t)}{\partial t} = -\int_0^t dt' Tr_B[H_I(t), [H_I(t'), \rho_S(t') \otimes \rho_B^{st}]] \tag{1.4.7}$$

Then we introduce the Markov approximation namely the interaction between system and environment is weak, or in other words, the environment is barely affected by coupling to system, which means that in the term $\rho_S(t')$ appeared in (1.4.7) we can replace

t' by t

$$\frac{\partial\rho_S(t)}{\partial t} = -\int_0^t dt' Tr_B[H_I(t);[H_I(t');\rho_S(t)\otimes\rho_B^{st}]] \tag{1.4.8}$$

Then in the term $H(t')$ we make the substitution changing t' to t

$$\frac{\partial\rho_S(t)}{\partial t} = -\int_0^t dt' Tr_B[H_I(t);[H_I(t-\tau);\rho_S(t)\otimes\rho_B^{st}]] \tag{1.4.9}$$

Taking time to the infinity, we can rewrite the above equation

$$\frac{\partial\rho_S(t)}{\partial t} = -\int_0^\infty dt' Tr_B[H_I(t);[H_I(t-\tau);\rho_S(t)\otimes\rho_B^{st}]] \tag{1.4.10}$$

1.5 Correlation functions

Let us continue to decompose H_I as

$$H_I = \sum_\alpha S_\alpha \otimes B_\alpha \tag{1.5.1}$$

where S_α and B_α represents the system operator and the environment operator, respectively. And the corresponding time-evolution Hamiltonian H_I as well as its components S_α and B_α can be expressed as

$$H_I(t) = \sum_\alpha S_\alpha(t) \otimes B_\alpha(t) \tag{1.5.2}$$

$$S_\alpha(t) = e^{iH_S t} S_\alpha e^{-iH_S t} \tag{1.5.3}$$

$$B_\alpha(t) = e^{iH_B t} B_\alpha e^{-iH_B t} \tag{1.5.4}$$

Expanding Equation (1.4.10)

$$\begin{aligned}\frac{\partial\rho_S(t)}{\partial t} = -\int_0^\infty dt' Tr_B\{&H_I(t)H_I(t-\tau)\rho_S(t)\otimes\rho_B^{st} \\ &- H_I(t-\tau)\rho_S(t)\otimes\rho_B^{st}H_I(t) - H_I(t)\rho_S(t)\otimes\rho_B^{st}H_I(t-\tau) \\ &+ \rho_S(t)\otimes\rho^{st}H_I(t-\tau)H_I(t)\}\end{aligned} \tag{1.5.5}$$

The correlation function is by definition a function that gives the statistical function between the variables, so the environment correlation function is given by the correlation of the environment operator with respect to the diﬀerent time denoted t_1 and t_2

$$C_{\alpha\beta}(t_1;t_2) = \langle B_\alpha(t_1)B_\beta(t_2)\rangle_B = Tr_B(B_\alpha(t_1)B_\beta(t_2)\rho_B) \tag{1.5.6}$$

As we have emphasized so many times, we can regard the environment as a stationary state. Here we need to point out that if the two variables refer to the same quantity from the function, this kind of correlation is called the autocorrelation function and therefore, the time variable in the second function can simply be moved to the first one in the form of the di□erence.

$$
\begin{aligned}
C\ \ (t_1; t_2) &= Tr_B(B\ \ (t_1\ \ \ t_2)B\ \ \ _B) \\
&= C\ \ (t_1\ \ \ t_2)
\end{aligned}
\tag{1.5.7}
$$

Inserting Equation (1.5.2) into Equation (1.5.5), and using the convention from Equation (1.5.7), Let us ignore the symbol , for the sake of simplicity

$$
\begin{aligned}
\frac{@\ \ _S(t)}{@t} = & \int_0^1 dt^0 Tr_B \sum S\ (t)\ \ B\ (t) \sum S\ (t\ \ \ )\ \ B\ (t\ \ \ )\ _S(t)\ \ \ ^{st}_B \\
& \sum S\ (t\ \ \ )\ \ B\ (t\ \ \ )\ _S(t)\ \ \ ^{st}_B \sum S\ (t)\ \ B\ (t) \sum S\ (t)\ \ B\ (t)\ _S(t)\ \ \ ^{st}_B \\
& \sum S\ (t\ \ \ )\ \ B\ (t\ \ \ ) + \ _S(t)\ \ \ ^{st}_B \sum S\ (t\ \ \ )\ \ B\ (t\ \ \ ) \sum S\ (t)\ \ B\ (t) \\
= & \int_0^1 dt^0 Tr_B \sum S\ (t); S\ (t\ \ \ )\ _S(t)\ C\ \ (\ \ ) + \ _S(t) S\ (t\ \ \ ); S\ (t)\ C\ \ (\ \ \ )
\end{aligned}
\tag{1.5.8}
$$

To apply the Markov approximation, note that the environment timescale should be shorter in comparison with the timescale of the system[7]. From the above equation, the environmental influence on the system state can be clearly showed.

Chapter 2

Quantum damping theory and quantum squeezed states

2.1 Quantum Langevin equations

In this section we shall focus more on a specific system interacting with a bath. The total Hamiltonian is in the form of

$$H = H_S + H_B + H_I \tag{2.1.1}$$

where H_S describes the Hamiltonian of the system S and H_B describes the Hamiltonian of the bath B. The interaction between S and B is represented by H_I. Ignoring the $\hbar$, in the case of harmonic oscillator corresponding to H_S, H_B and H_I are

$$\begin{aligned} H_S &= \omega_0 \hat{a}^\dagger \hat{a} \\ H_B &= \int_{-\infty}^{\infty} d\omega\, \omega \hat{b}^\dagger(\omega)\hat{b}(\omega) \\ H_I &= i\int_{-\infty}^{\infty} d\omega\, \kappa(\omega)\left[\hat{b}^\dagger \hat{a} - \hat{a}^\dagger \hat{b}(\omega)\right] \end{aligned} \tag{2.1.2}$$

where $\hat{a}$ and $\hat{a}^\dagger$ represent the corresponding system operators and $\hat{b}$ ($\hat{b}^\dagger$) is the boson annihilation (creation) operator; $\kappa(\omega)$ is the parameter with respect to ω. Substituting Equation (2.1.2) in the Heisenberg equation, we can obtain

$$\frac{\partial}{\partial t}\hat{b}(\omega) = -i\omega\hat{b}(\omega) + \kappa(\omega)\hat{a} \tag{2.1.3}$$

$$\frac{\partial}{\partial t}\hat{a} = -i\omega_0\hat{a} - \int_{-\infty}^{\infty} d\omega\, \kappa(\omega)\hat{b}(\omega) \tag{2.1.4}$$

Getting the solution from Equation (2.1.5)

$$\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + \kappa(\omega)\int_{t_0}^{t} e^{-i\omega(t-t')}a(t')dt' \tag{2.1.5}$$

Then substituting it in Equation (2.1.4)

$$\frac{\partial}{\partial t}\hat{a} = -i\omega_0\hat{a} - \int_{-\infty}^{\infty} d\omega\, \kappa(\omega) e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{t_0}^{t} d\omega [\kappa(\omega)]^2 e^{-i\omega(t-t')}\hat{a}(t') \tag{2.1.6}$$

The so called first Markov approximation[8] is proposed here

$$\kappa(\omega) = \sqrt{\frac{\gamma}{2\pi}} \tag{2.1.7}$$

Thus we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\hat{a} &= -i\omega_0\hat{a} - \int_{-\infty}^{\infty} d\omega \sqrt{\frac{\gamma}{2\pi}} e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{t_0}^{t} d\omega \frac{\gamma}{2\pi} e^{-i\omega(t-t')}\hat{a}(t') \\ &= -i\omega_0\hat{a} - \frac{\gamma}{2}\hat{a} - \sqrt{\gamma}\, b(t)\end{aligned} \tag{2.1.8}$$

where $b(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega)$, $\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} = 2\pi\delta(t-t')$ and $\int_{t_0}^{t}\hat{a}(t')\delta(t-t')dt' = \frac{1}{2}\hat{a}(t)$ are used[8]. In Equation (2.3.2) $\frac{1}{2}\gamma\hat{a}$ is the damping term.

2.2 Squeezed states and standard quantum limit

An electromagnetic field with a single mode consisting of a angular frequency ω, a wavevector k can be expressed as

$$E(t) = E_P sin(\omega t) + E_Q cos(\omega t) \tag{2.2.1}$$

E_P and E_Q are two quadrature operators

$$E_P = \varepsilon_0(\hat{a}_\omega + \hat{a}_\omega^\dagger) \tag{2.2.2}$$

$$E_Q = i\varepsilon_0(\hat{a}_\omega - \hat{a}_\omega^\dagger) \tag{2.2.3}$$

Where $\varepsilon_0 = \sqrt{\frac{\hbar\omega}{2\epsilon_0 cV}}$. Under the Heisenberg picture, the smallest quantum fluctuations is

$$\Delta E_P \Delta E_Q \geqslant \varepsilon_0^2 \tag{2.2.4}$$

The case when $\Delta E_P = \Delta E_Q = \varepsilon_0$ is called standard quantum limit (SQL) or short-noise limit. By reducing one of the quadrature below the SQL, the squeezed state is then obtained. In the vacuum, the quantum fluctuations still exist. One evidence to prove this quantum fluctuations is the spontaneous emission phenomena. The definition for the two quadrature operators given above is not the general case. Considering the phase difference θ

$$E_P = \varepsilon_0(\hat{a}_\omega e^{i\theta} + \hat{a}^\dagger_\omega e^{-i\theta}) \tag{2.2.5}$$

$$E_Q = i\varepsilon_0(\hat{a}_\omega e^{i\theta} - \hat{a}^\dagger_\omega e^{-i\theta}) \tag{2.2.6}$$

Actually the new quadrature operators with phase difference term θ still satisfy the inequality equation (2.2.4). Such non-classical states are called squeezed states.

2.3 Quadrature variance

The so called quadrature amplitude is defined as

$$\hat{X}_\theta = \hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta} \tag{2.3.1}$$

$$\langle \hat{X}_\theta \rangle = \langle \hat{a} \rangle e^{-i\theta} + \langle \hat{a}^\dagger \rangle e^{i\theta} \tag{2.3.2}$$

$$\langle \hat{X}_\theta \rangle^2 = \langle \hat{a} \rangle^2 e^{-2i\theta} + \langle \hat{a} \rangle\langle \hat{a}^\dagger \rangle + \langle \hat{a}^\dagger \rangle\langle \hat{a} \rangle + \langle \hat{a}^\dagger \rangle^2 e^{2i\theta} \tag{2.3.3}$$

$$\begin{aligned} \langle \hat{X}_\theta^2 \rangle &= \langle \hat{a}^2 e^{-2i\theta} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}^{\dagger 2} e^{2i\theta} \rangle \\ &= \langle \hat{a}^2 \rangle e^{-2i\theta} + \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^\dagger\hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle e^{2i\theta} \end{aligned} \tag{2.3.4}$$

The variance of this quadrature amplitude is given by

$$\begin{aligned} \Delta\hat{X}_{\theta\theta'}^2 =& \langle \hat{X}_\theta^2 \rangle - \langle \hat{X}_{\theta'} \rangle^2 \\ =& \langle \hat{a}^2 \rangle e^{-2i\theta} + \langle \hat{a}\hat{a}^\dagger \rangle + \langle \hat{a}^\dagger\hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle e^{2i\theta} - \langle \hat{a} \rangle^2 e^{-2i\theta'} - \langle \hat{a} \rangle\langle \hat{a}^\dagger \rangle - \langle \hat{a}^\dagger \rangle\langle \hat{a} \rangle \\ & - \langle \hat{a}^\dagger \rangle^2 e^{2i\theta'} \\ =& \langle \hat{a}^2 \rangle e^{-2i\theta} - \langle \hat{a} \rangle^2 e^{-2i\theta'} + \langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a} \rangle\langle \hat{a}^\dagger \rangle + \langle \hat{a}^\dagger\hat{a} \rangle - \langle \hat{a}^\dagger \rangle\langle \hat{a} \rangle + \langle \hat{a}^{\dagger 2} \rangle e^{2i\theta} \\ & - \langle \hat{a}^\dagger \rangle^2 e^{2i\theta'} \end{aligned} \tag{2.3.5}$$

Denoting $\theta' = \theta + \phi$ to distinguish two different angles

$$\begin{aligned} \Delta\hat{X}_{\theta\theta'}^2 =& \left(\langle \hat{a}^2 \rangle e^{2i\phi} - \langle \hat{a} \rangle^2\right) e^{-2i(\theta+\phi)} + \left(\langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a} \rangle\langle \hat{a}^\dagger \rangle\right) + \left(\langle \hat{a}^\dagger\hat{a} \rangle - \langle \hat{a}^\dagger \rangle\langle \hat{a} \rangle\right) \\ & + \left(\langle \hat{a}^{\dagger 2} \rangle e^{-2i\phi} - \langle \hat{a}^\dagger \rangle^2\right) e^{2i(\theta+\phi)} \end{aligned} \tag{2.3.6}$$

Ususally we take $\theta = 0$ for simplicity, we thus obtain

$$\left(\Delta \hat{X}_{00}\right)^2 = \left(\langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2\right) e^{-2i} + \langle \hat{a}\hat{a}^\dagger \rangle - \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle + \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle + \left(\langle \hat{a}^{\dagger 2} \rangle - \langle \hat{a}^\dagger \rangle^2\right) e^{2i} \tag{2.3.7}$$

Then substituting $\hat{a}_0 = \langle \hat{a}_0 \rangle + \Delta \hat{a}_0$ into the above-mentioned equation

$$\left(\Delta \hat{X}_{;0}\right)^2 = \left(\langle (\Delta \hat{a}_0)^2 \rangle - \langle \Delta \hat{a}_0 \rangle^2\right) e^{-2i} + \langle \Delta \hat{a}_0^\dagger \Delta \hat{a}_0 \rangle + \langle \Delta \hat{a}_0 \Delta \hat{a}_0^\dagger \rangle - \langle \Delta \hat{a}_0^\dagger \rangle \langle \Delta \hat{a}_0 \rangle - \langle \Delta \hat{a}_0 \rangle \langle \Delta \hat{a}_0^\dagger \rangle + \left(\langle (\Delta \hat{a}_0^\dagger)^2 \rangle - \langle \Delta \hat{a}_0^\dagger \rangle^2\right) e^{2i} \tag{2.3.8}$$

The Equation (2.3.8) shows the full formula to calculate this variance representing how many degrees the light is squeezed. In coherent state $\langle \Delta \hat{a}_0^\dagger \rangle \langle \Delta \hat{a}_0 \rangle$, $\langle \Delta \hat{a}_0 \rangle \langle \Delta \hat{a}_0^\dagger \rangle$, $\langle \Delta \hat{a}_0 \rangle^2$ and $\langle \Delta \hat{a}_0^\dagger \rangle^2$shall be zero, so the equation can be simplified

$$\left(\Delta \hat{X}_{;0}\right)^2 = \langle (\Delta \hat{a}_0)^2 \rangle e^{-2i} + \langle (\Delta \hat{a}_0^\dagger)^2 \rangle e^{2i} + \langle \Delta \hat{a}_0^\dagger \Delta \hat{a}_0 \rangle + \langle \Delta \hat{a}_0 \Delta \hat{a}_0^\dagger \rangle \tag{2.3.9}$$

A squeezed coherent state is therefore an ideal squeezed state[9].

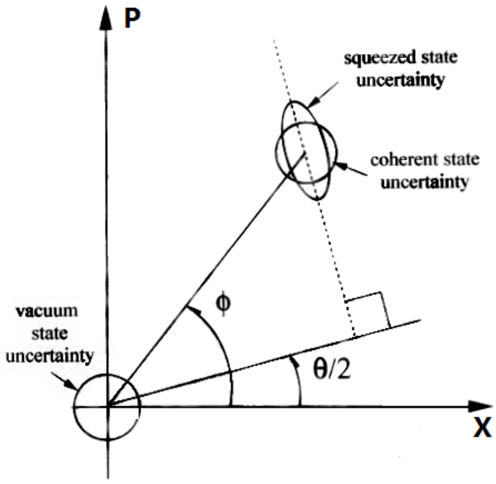

Figure 2.1: The uncertainty areas in the generalized coordinate and momentum (X, P) phase space of a photon squeezed state (ellipse) and a coherent state (circle). Image Source: X. Hu, Quantum Fluctuations In Condensed Matter Systems, UM Ph.D. Thesis 1997, Page 5.

Notice that the coherent state has the same uncertainty area as the vacuum state (circle at the origin), and that its area is circular, while the squeezed state has an elliptical uncertainty area[9].

Chapter 3

Non-linear Induced Decoherence

3.1 2D photonic system

Let us consider a photonic system consisting of mode of frequency ω_i. As such a system, the Hamiltonian of this system is given by

$$H = \sum_i \hbar \omega_i \hat{a}_i^\dagger \hat{a}_i + \frac{\hbar}{2} \sum_{ijkl} g_{ijkl} \hat{a}_i^\dagger \hat{a}_j^\dagger \hat{a}_k \hat{a}_l \tag{3.1.1}$$

Note that in this case the Hamiltonian does not contain the environment part and it shall be treated later as the dissipation term considered in the dynamics equation. ω_i and $\hat{a}_i$ are the eigenfrequency and boson annihilation operator of the i-th photon eigenmode and the quantity g_{ijkl} is the nonlinear coupling coefficient with different states from ground state (0-th) to the higher (n-th) state.

Considering the dynamics of the system interacting with the environment. The quantum Langevin equations is introduced in Equation (2.3.2) satisfying

$$\frac{\partial}{\partial t} \hat{a}_n = -\frac{i}{\hbar} [\hat{a}_n, H] - \frac{\gamma_n}{2} \hat{a}_n + \hat{F}_n e^{-i\omega_p t} \tag{3.1.2}$$

where γ_n is the cavity damping constant of n-th modes; the last term is the pumping term in which the ω_p represents the frequency of the leaser produced from the optical pump. The electronic field generated by the laser is expressed under the semi-classical approach. In (3.1.2), the first term is the Hamiltonian part and the second one is about the dissipation. Using Equation (3.1.1) in Equation (3.1.2) reads

$$\frac{\partial}{\partial t} \hat{a}_n = -i\omega_n \hat{a}_n - i\frac{1}{2} \sum_{jkl} g_{njkl} \hat{a}_j^\dagger \hat{a}_k \hat{a}_l - i\frac{1}{2} \sum_{ikl} g_{inkl} \hat{a}_i^\dagger \hat{a}_k \hat{a}_l - \frac{\gamma_n}{2} \hat{a}_n + \hat{F}_n e^{-i\omega_p t} \tag{3.1.3}$$

Substituting $\hat{a}_n$ by its mean field and notice that $\hat{a}_n = \langle \hat{a}_n \rangle e^{-i\omega_p t}$ without considering fluctuation here in rotating frame with considering the optical pumping of the coupling of the cavity. If only the mode is coherently excited in the ground state, in other words that only when $n = 0$. At this stage if F disappears is because $\langle \hat{F} \rangle = 0$

$$\frac{\partial}{\partial t}\langle \hat{a}_0 \rangle = -i\omega_{0p}\langle \hat{a}_0 \rangle - ig_{0000}\langle \hat{a}_0^\dagger \rangle\langle \hat{a}_0 \rangle\langle \hat{a}_0 \rangle - \frac{\gamma_0}{2}\langle \hat{a}_0 \rangle + \langle \hat{F}_0 \rangle \tag{3.1.4}$$

where $\omega_{0p} = \omega_0 - \omega_p$. Recalling the occupied stated number defined as $n = \hat{a}^\dagger \hat{a}$, the following result can be expressed as

$$\langle n_0 \rangle = \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \approx \langle \hat{a}_0^\dagger \rangle\langle \hat{a}_0 \rangle \tag{3.1.5}$$

where $\langle n_0 \rangle$ shows the mean value of the occupied number at ground state, we thus obtain

$$\frac{\partial}{\partial t}\langle \hat{a}_0 \rangle = -i\omega_{0p}\langle \hat{a}_0 \rangle - ig_{0000}\langle n_0 \rangle\langle \hat{a}_0 \rangle - \frac{\gamma_0}{2}\langle \hat{a}_0 \rangle + \langle \hat{F}_0 \rangle \tag{3.1.6}$$

To find the steady-state by taking $\frac{\partial}{\partial t}\langle \hat{a}_0 \rangle = 0$

$$-i\omega_{0p}\langle \hat{a}_0 \rangle - ig_{0000}\langle n_0 \rangle\langle \hat{a}_0 \rangle - \frac{\gamma_0}{2}\langle \hat{a}_0 \rangle + \langle \hat{F}_0 \rangle = 0 \tag{3.1.7}$$

To derive the intensity I_0 of the leaser from the above equations, taking the mean field approximation and the modules of the square of the $\langle \hat{F}_0 \rangle$

$$I_0 = \langle \hat{F}_0 \rangle^2 = \left(\frac{\gamma^2}{4} + \omega_{0p}^2\right) n_0 + 2g_{0000}\,\omega_{0p} n_0^2 + g_{0000}^2 n_0^3 \tag{3.1.8}$$

Fixing several parameters, we can get the plot showing the intensity of the leaser with increasing the photon numbers

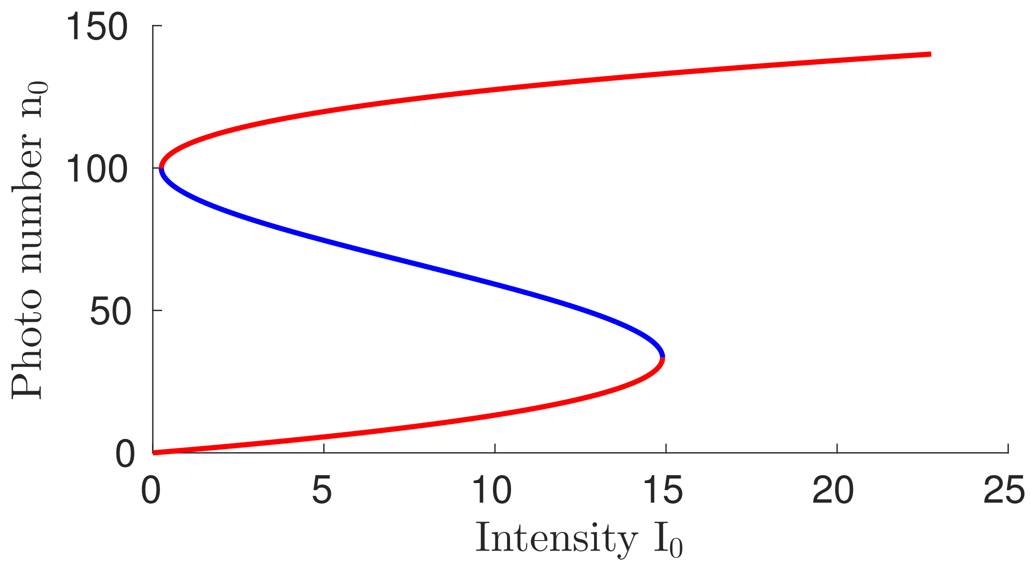

Figure 3.1: Red line: The intensity versus the photo number n_0. Blue line: The unstable region. Parameters: $g_{0000} = 0.01$meV m^2; ${}_{0} = 0.1$meV; $\omega_{0p} = $ 1meV

The intensity plot shows there exists a unstable region in which the slop of the curve changes dramatically with increasing n_0 and the two edge points holding the maximum value of the slop correspond to respective the two squeezing points, which shall be discussed in the following section.

3.2 The evolution of the fluctuation in single mode

Considering the fluctuation, the mean value of the operator $\hat{a}_n$ thus can approximately be replaced by the steady-state. Note that $\hat{a}_n$ and $\delta\hat{a}_n$ are time-dependent, but $\langle\hat{a}_0\rangle^{st}$ is time-independent

$$\hat{a}_0(t) = \langle\hat{a}_0(t)\rangle + \delta\hat{a}_0(t) \approx \langle\hat{a}_0\rangle_{st} + \delta\hat{a}_0(t) \tag{3.2.1}$$

Taking the substitution of Equation (3.2.1) into (3.1.4) and using mean field approximation. But here we do not ignore the fluctuation of $\hat{F}_0$ namely $\delta\hat{F}_0 \neq 0$

$$
\begin{aligned}
\frac{\partial}{\partial t}\hat{a}_0(t) = & -i\omega_{0p}(\langle\hat{a}_0\rangle_{st} + \delta\hat{a}_0) - ig_{0000}(\langle\hat{a}_0\rangle_{st} + \delta\hat{a}_0)^\dagger(\langle\hat{a}_0\rangle_{st} + \delta\hat{a}_0)(\langle\hat{a}_0\rangle_{st} + \delta\hat{a}_0) \\
& + \langle\hat{F}_0\rangle + \delta\hat{F}_0 - \frac{\kappa_0}{2}(\langle\hat{a}_0\rangle_{st} + \delta\hat{a}_0)
\end{aligned} \tag{3.2.2}
$$

Expanding the equation

$$
\begin{aligned}
\frac{\partial}{\partial t}\delta\hat{a}_0 = & -i\omega_{0p}\langle\hat{a}_0\rangle_{st} - ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st} - \frac{\kappa_0}{2}\langle\hat{a}_0\rangle_{st} + \langle\hat{F}_0\rangle \\
& + \delta\hat{F}_0 - i\omega_{0p}\delta\hat{a}_0 - \frac{\kappa_0}{2}\delta\hat{a}_0 - ig_{0000}\Big(2\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_0 \\
& + \langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_0^\dagger + 2\langle\hat{a}_0\rangle_{st}\delta\hat{a}_0^\dagger\delta\hat{a}_0 + \langle\hat{a}_0^\dagger\rangle_{st}\delta\hat{a}_0\delta\hat{a}_0 + \delta\hat{a}_0^\dagger\delta\hat{a}_0\delta\hat{a}_0\Big)
\end{aligned} \tag{3.2.3}
$$

The first line of the formula is canceled because it is in steady-state, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\delta\hat{a}_0 = & -i\omega_{0p}\delta\hat{a}_0 - \frac{\kappa_0}{2}\delta\hat{a}_0 + \delta\hat{F}_0 - ig_{0000}\Big(2\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_0 \\
& + \langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_0^\dagger + 2\langle\hat{a}_0\rangle_{st}\delta\hat{a}_0^\dagger\delta\hat{a}_0 + \langle\hat{a}_0^\dagger\rangle_{st}\delta\hat{a}_0\delta\hat{a}_0 + \delta\hat{a}_0^\dagger\delta\hat{a}_0\delta\hat{a}_0\Big)
\end{aligned} \tag{3.2.4}
$$

If we ignore the second and third order term and put the H.c. of Equation (3.2.4)

$$
\frac{\partial}{\partial t}\delta\hat{a}_0 = \left(-i\omega_{0p} - \frac{\kappa_0}{2} - 2ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\right)\delta\hat{a}_0 - ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_0^\dagger + \delta\hat{F}_0 \tag{3.2.5}
$$

$$
\frac{\partial}{\partial t}\delta\hat{a}_0^\dagger = \left(i\omega_{0p} - \frac{\kappa_0}{2} + 2ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st}\right)\delta\hat{a}_0^\dagger + ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st}\delta\hat{a}_0 + \delta\hat{F}_0^\dagger \tag{3.2.6}
$$

Rewriting it in the form of the matrix

$$
\frac{\partial}{\partial t}\begin{pmatrix}\delta\hat{a}_0 \\ \delta\hat{a}_0^\dagger\end{pmatrix} = \begin{pmatrix} -i\omega_{0p} - \frac{\kappa_0}{2} - 2ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st} & -ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st} \\ ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st} & i\omega_{0p} - \frac{\kappa_0}{2} + 2ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st}\end{pmatrix}\begin{pmatrix}\delta\hat{a}_0 \\ \delta\hat{a}_0^\dagger\end{pmatrix} \tag{3.2.7}
$$

Denote the right-hand matrix M and find the eigenvalue from the equation det[M $-\lambda$I] = 0, we obtain the eigenvalue

$$
\lambda = \frac{1}{2}\Big[-i\kappa_0 \pm 2\Big(-\omega_{0p}^2 + 4g_{0000}\,\omega_{0p}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st} + 3g_{0000}^2\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\Big)^{\frac{1}{2}}\Big] \tag{3.2.8}
$$

where the mean field approximation namely $\langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle \approx \langle\hat{a}^\dagger\hat{a}\rangle = \langle n\rangle \approx n$ is already considered.

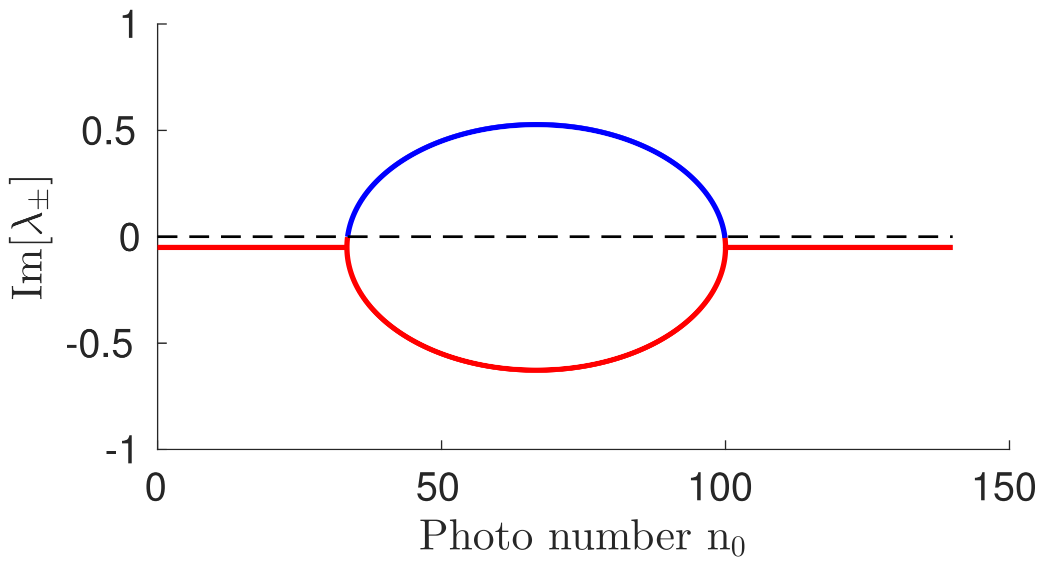

Figure 3.2: Red line: The imaginary part of the eigenvalue is smaller than 0. Blue line: The part greater than 0. Dashed line: Datum line in which the eigenvalue is equal to 0. Parameters: $g_{0000} = 0{:}01meV \quad m^2; \quad _0 = 0{:}1meV; \, !_{0p} = \quad 1$meV

The imaginary part of the eigenvalue which is greater than 0 means the system is in unstable condition, furthermore, the two points where eigenvalue equals to 0 is right the turning points appearing in Figure 3.1

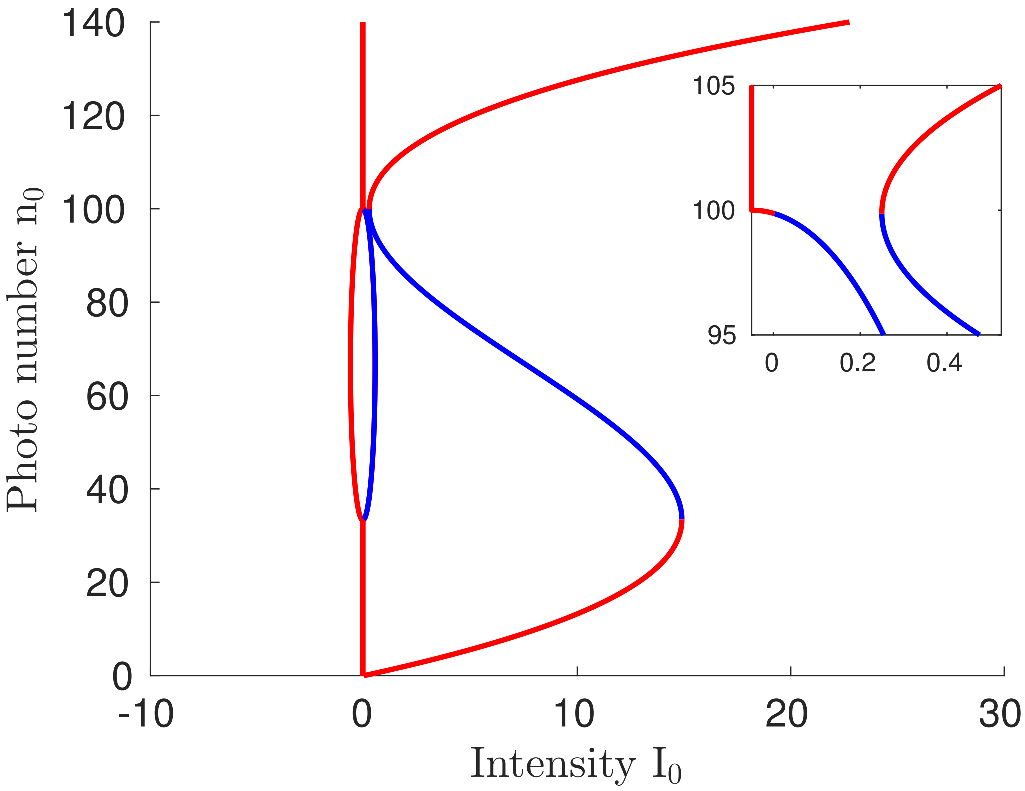

Figure 3.3: Red line: The imignary part of eigenvalue smaller than 0. Blue line: The imignary part of eigenvalue greater than 0. Dashed line: Datum line in which eigenvalue is equal to 0. Parameters: $g_{0000} = 0.01\mathrm{meV}\ \mathrm{m}^2; \gamma_0 = 0.1\mathrm{meV}$

The Figure 3.3 clearly shows that the both blue regions coincide but we still need to note that the intensity we calculated is just the background without considering the fluctuation. Although the approximation made by neglecting the high order term seems not strict, here we do not much care the influence on the fluctuation.

We now want to determine the variance of the quadrature as defined in Equation (2.3.8). To this purpose we need to know $\langle \hat{a}_0^2 \rangle$. Looking at Equation (3.2.7) In terms of frequency components, $\hat{a}_0$ is defined by Fourier transform[10]

$$\tilde{a}_0(\omega) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} \hat{a}_0(t) dt \tag{3.2.9}$$

Using it in ?? kills the differential part and reduces to

$$\begin{pmatrix} i(\omega_{0p} - \omega) + \frac{\gamma_0}{2} + 2ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st} & +ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st} \\ -ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st} & -i(\omega_{0p} + \omega) + \frac{\gamma_0}{2} - 2ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st} \end{pmatrix} \begin{pmatrix} \tilde{a}_0(\omega) \\ \tilde{a}_0^\dagger(-\omega) \end{pmatrix} = \begin{pmatrix} \tilde{F}_0(\omega) \\ \tilde{F}_0^\dagger(-\omega) \end{pmatrix} \tag{3.2.10}$$

To express $\Delta\hat{a}_0(\omega)$ and $\Delta\hat{a}_0^\dagger(-\omega)$ in terms of $\hat{F}_0(\omega)$ and $\hat{F}_0^\dagger(-\omega)$, the inverse matrix should be taken

$$\begin{pmatrix}\Delta\hat{a}_0(\omega)\\ \Delta\hat{a}_0^\dagger(-\omega)\end{pmatrix} = \frac{1}{\frac{\kappa_0^2}{4}+3g_{0000}^2\langle\hat{a}_0^\dagger\rangle_{st}^2\langle\hat{a}_0\rangle_{st}^2-i\kappa_0\omega-\omega^2+4g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\omega_{0p}+\omega_{0p}^2}$$
$$\begin{pmatrix}\frac{\kappa_0}{2}-2ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}-i(\omega+\omega_{0p}) & -ig_{0000}\langle\hat{a}_0\rangle_{st}^2\\ ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}^2 & \frac{\kappa_0}{2}+2ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}+i(-\omega+\omega_{0p})\end{pmatrix}\begin{pmatrix}\hat{F}_0(\omega)\\ \hat{F}_0^\dagger(-\omega)\end{pmatrix} \tag{3.2.11}$$

To calculate $\langle\Delta\hat{a}_0(\omega);\Delta\hat{a}_0(\omega')\rangle$, note that the condition $\langle\hat{F}_0(\omega)\rangle = \langle\hat{F}_0^\dagger(\omega')\rangle = 0$ and $\langle\hat{F}_0\hat{F}_0^\dagger\rangle = \kappa_0$ have been used and recalling Equation (2.3.9) $\langle(\Delta\hat{X}_{\theta,0})^2\rangle = \langle(\Delta\hat{a}_0)^2\rangle e^{-2i\theta} + \langle\Delta\hat{a}_0^\dagger\Delta\hat{a}_0\rangle + \langle\Delta\hat{a}_0\Delta\hat{a}_0^\dagger\rangle + \langle(\Delta\hat{a}_0^\dagger)^2\rangle e^{2i\theta}$ in Chapter 2

$$\langle(\Delta\hat{X}_{\theta,0})^2\rangle = \frac{1}{12g_{0000}^2\langle\hat{a}_0^\dagger\rangle_{st}^2\langle\hat{a}_0\rangle_{st}^2+\kappa_0^2+4\omega_{0p}\left(4g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}+\omega_{0p}\right)}$$
$$\Big[e^{-2i\theta}\Big(e^{4i\theta}g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}^2\left(4g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}-i\kappa_0+2\omega_{0p}\right)-g_{0000}\langle\hat{a}_0\rangle_{st}^2\left(4g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\right.$$
$$\left.+i\kappa_0+2\omega_{0p}\right)\Big)+e^{2i\theta}\Big(16g_{0000}^2\langle\hat{a}_0^\dagger\rangle_{st}^2\langle\hat{a}_0\rangle_{st}^2+\kappa_0^2+4\omega_{0p}\left(4g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}+\omega_{0p}\right)\Big)\Big] \tag{3.2.12}$$

as we have expected, when the photo number increase to a vary number meanwhile the variance of the quadrature reaches the minimal value which is around 0.5[11], recalling Figure 3.3, the intensity begins traveling the unstable region. Increasing the photo number continually, the minimal property appears again, which means that stability of intensity is held again. Furthermore, that two minimal points are respective the two turning points appeared in Figure 3.3. A more clear result is shown in Figure 3.5: with changing the parameter $\langle\hat{a}_0\rangle_{st}$ and $\langle\hat{a}_0^\dagger\rangle_{st}$, namely changing the photo number, the minimal value of the variance would also change.

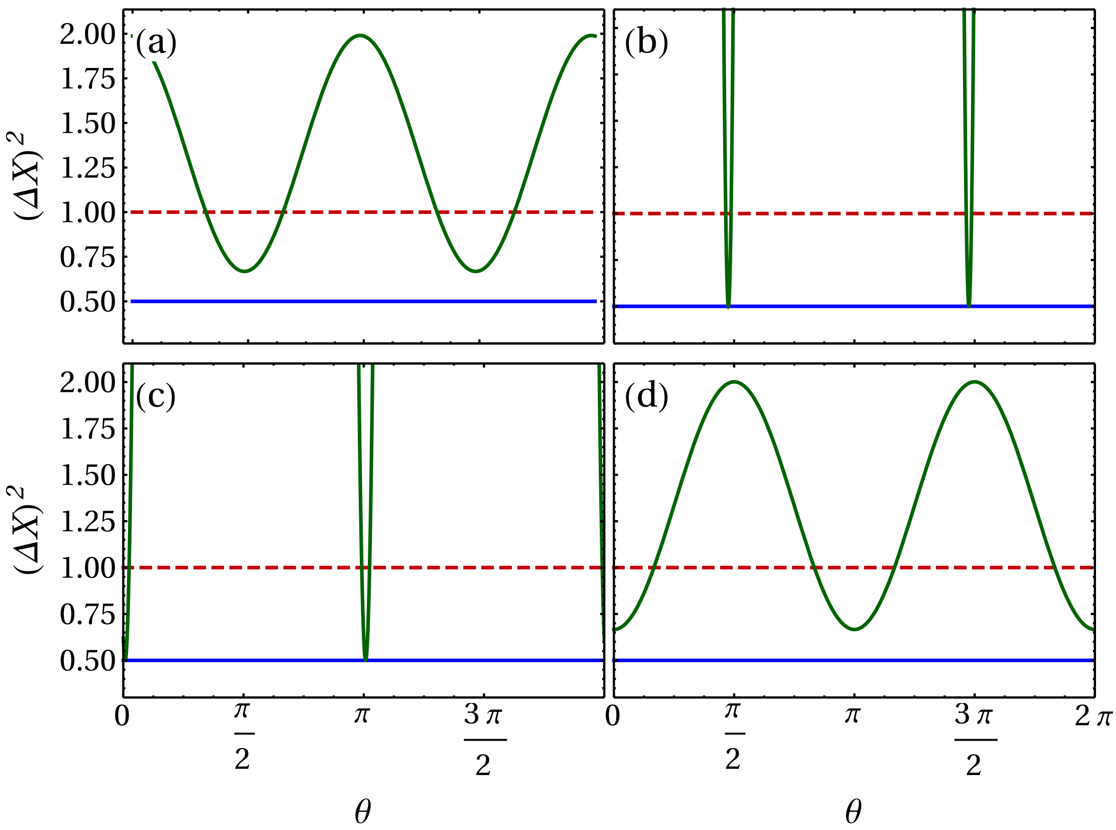

Figure 3.4: Red dashed line: Standard quantum limit. Blue line: The minimal value that the squeezing can reach. Green line: The variance of quadrature operator versus angle . (a) the corresponding intensity is in stable region. (b) the corresponding intensity is in unstable region meanwhile the variance reaches the minimal value. (c) the corresponding intensity is still in unstable region meanwhile the variance reaches the minimal value. (d) the corresponding intensity is in stable region again. Parameters: (a)$h\hat{a}_0 i_{st} = h\hat{a}_0^y i_{st} = 5(b)h\hat{a}_0 i_{st} = h\hat{a}_0^y i_{st} = 5{:}782(c)h\hat{a}_0 i_{st} = h\hat{a}_0^y i_{st} = 10{:}2$ (d)$h\hat{a}_0 i_{st} = h\hat{a}_0^y i_{st} = 300$. $g_{0000} = 0{:}01\mathrm{meV}\ \ \mathrm{m}^2;\ \ _0 = 0{:}1\mathrm{meV}$

Up to now, we have a self-consistent theory to explain the relationship between the intensity and the photo number, however, the accuracy of the variance of the fluctuation $\hat{a}_0$ becomes a question that should we neglect the high order term of the fluctuation? If not, how many contribution should these terms give?

3.3 The evolution of the fluctuation in multimode

There are several ways to take the high order terms into account. For example, we can calculate the fluctuation of the $\hat{a}_0$, by following the procedure in last section, a result could be got but less meaningful, because the fluctuation is already very small. We just limit ourselves to the single mode, with considering the multimode instead, the higher order of the fluctuation from the another modes surly gives a clear clue to solve the problem.

Back to Equation (3.1.3), the full expansion of $\hat{a}_n = (\langle\hat{a}_0\rangle_{st} + \delta\hat{a}_n)e^{-i\omega_p t}$ instead of the only meanfield and $\hat{F}_n = \langle\hat{F}_n\rangle + \delta\hat{F}_n$ is considered.

$$\begin{aligned}
\frac{\partial}{\partial t}\langle\hat a_n\rangle + \frac{\partial}{\partial t}\delta\hat a_n = & -i(\omega_n-\omega_p)\langle\hat a_n\rangle - i\frac{1}{2}\sum_{jkl} g_{njkl}\langle\hat a_j^\dagger\rangle\langle\hat a_k\rangle\langle\hat a_l\rangle - i\frac{1}{2}\sum_{ikl} g_{inkl}\langle\hat a_i^\dagger\rangle\langle\hat a_k\rangle\langle\hat a_l\rangle \\
& -\frac{\kappa_n}{2}\langle\hat a_n\rangle + \langle\hat F_n\rangle - i(\omega_n-\omega_p)\delta\hat a_n - i\frac{1}{2}\sum_{jkl} g_{njkl}\Big(\langle\hat a_j^\dagger\rangle\langle\hat a_k\rangle\delta\hat a_l + \langle\hat a_k\rangle\langle\hat a_l\rangle\delta\hat a_j^\dagger \\
& + \langle\hat a_j^\dagger\rangle\langle\hat a_l\rangle\delta\hat a_k + \langle\hat a_l\rangle\delta\hat a_j^\dagger\delta\hat a_k + \langle\hat a_j^\dagger\rangle\delta\hat a_k\delta\hat a_l + \langle\hat a_k\rangle\delta\hat a_j^\dagger\delta\hat a_l + \delta\hat a_j^\dagger\delta\hat a_k\delta\hat a_l\Big) \\
& - i\frac{1}{2}\sum_{ikl} g_{inkl}\Big(\langle\hat a_i^\dagger\rangle\langle\hat a_k\rangle\delta\hat a_l + \langle\hat a_k\rangle\langle\hat a_l\rangle\delta\hat a_i^\dagger + \langle\hat a_i^\dagger\rangle\langle\hat a_l\rangle\delta\hat a_k + \langle\hat a_l\rangle\delta\hat a_i^\dagger\delta\hat a_k + \langle\hat a_i^\dagger\rangle\delta\hat a_k\delta\hat a_l \\
& + \langle\hat a_k\rangle\delta\hat a_i^\dagger\delta\hat a_l + \delta\hat a_i^\dagger\delta\hat a_k\delta\hat a_l\Big) - \frac{\kappa_n}{2}\delta\hat a_n + \delta\hat F_n
\end{aligned} \tag{3.3.1}$$

Taking the following condition

$$\begin{aligned}
\hat F_n &= \delta_{n0}\hat F_0 + \delta\hat F_n \\
\langle\hat a_n\rangle &= \delta_{n0}\langle\hat a_0\rangle_{st} \\
\langle\hat a_n^\dagger\rangle &= \delta_{n0}\langle\hat a_0^\dagger\rangle_{st}
\end{aligned} \tag{3.3.2}$$

where $\langle\hat a_0\rangle_{st}$ and $\langle\hat a_0^\dagger\rangle_{st}$ mean the steady-state solution.

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\delta_{n0}\langle\hat a_0\rangle_{st}\right) + \frac{\partial}{\partial t}\delta\hat a_n = & -i(\omega_n-\omega_p)\,\delta_{n0}\langle\hat a_0\rangle_{st} - i\frac{1}{2}\sum_{jkl} g_{njkl}\,\delta_{j0}\delta_{k0}\delta_{l0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\langle\hat a_0\rangle_{st} \\
& - i\frac{1}{2}\sum_{ikl} g_{inkl}\,\delta_{i0}\delta_{k0}\delta_{l0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\langle\hat a_0\rangle_{st} - \frac{\kappa_n}{2}\delta_{n0}\langle\hat a_0\rangle_{st} + \delta_{n0}\langle\hat F_0\rangle - i(\omega_n-\omega_p)\,\delta\hat a_n \\
& - i\frac{1}{2}\sum_{jkl} g_{njkl}\Big(\delta_{j0}\delta_{k0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\,\delta\hat a_l + \delta_{k0}\delta_{l0}\langle\hat a_0\rangle_{st}\langle\hat a_0\rangle_{st}\,\delta\hat a_j^\dagger + \delta_{j0}\delta_{l0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\,\delta\hat a_k \\
& + \delta_{l0}\langle\hat a_0\rangle_{st}\,\delta\hat a_j^\dagger\delta\hat a_k + \delta_{j0}\langle\hat a_0^\dagger\rangle_{st}\,\delta\hat a_k\delta\hat a_l + \delta_{k0}\langle\hat a_0\rangle_{st}\,\delta\hat a_j^\dagger\delta\hat a_l + \delta\hat a_j^\dagger\delta\hat a_k\delta\hat a_l\Big) \\
& - i\frac{1}{2}\sum_{ikl} g_{inkl}\Big(\delta_{i0}\delta_{k0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\,\delta\hat a_l + \delta_{k0}\delta_{l0}\langle\hat a_0\rangle_{st}\langle\hat a_0\rangle_{st}\,\delta\hat a_i^\dagger + \delta_{i0}\delta_{l0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\,\delta\hat a_k \\
& + \delta_{l0}\langle\hat a_0\rangle_{st}\,\delta\hat a_i^\dagger\delta\hat a_k + \delta_{i0}\langle\hat a_0^\dagger\rangle_{st}\,\delta\hat a_k\delta\hat a_l + \delta_{k0}\langle\hat a_0\rangle_{st}\,\delta\hat a_i^\dagger\delta\hat a_l + \delta\hat a_i^\dagger\delta\hat a_k\delta\hat a_l\Big) - \frac{\kappa_n}{2}\delta\hat a_n + \delta\hat F_n
\end{aligned} \tag{3.3.3}$$

Taking $n = 0$ and notice that the sum terms vanished by suming over the δ_{l0} and δ_{k0}, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\langle\hat{a}_0\rangle_{st} + \frac{\partial}{\partial t}\delta\hat{a}_0 = & -i\omega_{0p}\langle\hat{a}_0\rangle_{st} - ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st} - \frac{\gamma_0}{2}\langle\hat{a}_0\rangle_{st} + \langle\hat{F}_0\rangle - i\omega_{0p}\delta\hat{a}_0 \\
& - i\left(\sum_l g_{000l}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_l + \sum_k g_{00k0}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_k\right) - i\frac{1}{2}\left(\sum_i g_{i000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_i^\dagger\right. \\
& \left. + \sum_j g_{0j00}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_j^\dagger\right) - i\frac{1}{2}\left(\sum_{jk} g_{0jk0}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_j^\dagger\delta\hat{a}_k + \sum_{il} g_{i00l}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_i^\dagger\delta\hat{a}_l\right. \\
& \left. + \sum_{ik} g_{i0k0}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_i^\dagger\delta\hat{a}_k + \sum_{jl} g_{0j0l}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_j^\dagger\delta\hat{a}_l\right) - i\sum_{kl} g_{00kl}\langle\hat{a}_0^\dagger\rangle_{st}\delta\hat{a}_k\delta\hat{a}_l \\
& - i\frac{1}{2}\left(\sum_{ikl} g_{i0kl}\delta\hat{a}_i^\dagger\delta\hat{a}_k\delta\hat{a}_l + \sum_{jkl} g_{0jkl}\delta\hat{a}_j^\dagger\delta\hat{a}_k\delta\hat{a}_l\right) \\
& - \frac{\gamma_0}{2}\delta\hat{a}_0 + \delta\hat{F}_0
\end{aligned}
\tag{3.3.4}
$$

The steady-state prat vanished, thus we obtain

$$
\begin{aligned}
\frac{\partial}{\partial t}\delta\hat{a}_0 = & -\left(i\omega_{0p} + \frac{\gamma_0}{2}\right)\delta\hat{a}_0 - i\left(\sum_l g_{000l}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_l + \sum_k g_{00k0}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_k\right) \\
& - i\frac{1}{2}\left(\sum_i g_{i000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_i^\dagger + \sum_j g_{0j00}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_j^\dagger\right) - i\frac{1}{2}\left(\sum_{jk} g_{0jk0}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_j^\dagger\delta\hat{a}_k\right. \\
& \left. + \sum_{il} g_{i00l}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_i^\dagger\delta\hat{a}_l + \sum_{ik} g_{i0k0}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_i^\dagger\delta\hat{a}_k + \sum_{jl} g_{0j0l}\langle\hat{a}_0\rangle_{st}\delta\hat{a}_j^\dagger\delta\hat{a}_l\right) \\
& - i\sum_{kl} g_{00kl}\langle\hat{a}_0^\dagger\rangle_{st}\delta\hat{a}_k\delta\hat{a}_l - i\frac{1}{2}\left(\sum_{ikl} g_{i0kl}\delta\hat{a}_i^\dagger\delta\hat{a}_k\delta\hat{a}_l + \sum_{jkl} g_{0jkl}\delta\hat{a}_j^\dagger\delta\hat{a}_k\delta\hat{a}_l\right) + \delta\hat{F}_0
\end{aligned}
\tag{3.3.5}
$$

In the single mode case, comparing this result with Equation (3.2.4), the one to one correspondence is shown as follows

$$
\begin{aligned}
i2g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\,\hat{a}_0 &\to i\left(\sum_l g_{000l}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\,\hat{a}_l+\sum_k g_{00k0}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}\,\hat{a}_k\right)\\
ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\,\hat{a}_0^\dagger &\to i\frac{1}{2}\left(\sum_i g_{i000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\,\hat{a}_i^\dagger+\sum_j g_{0j00}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\,\hat{a}_j^\dagger\right)\\
i2g_{0000}\langle\hat{a}_0\rangle_{st}\,\hat{a}_0^\dagger\,\hat{a}_0 &\to \begin{cases} i\frac{1}{2}\left(\sum_{jk} g_{0jk0}\langle\hat{a}_0\rangle_{st}\,\hat{a}_j^\dagger\,\hat{a}_k+\sum_{il} g_{i00l}\langle\hat{a}_0\rangle_{st}\,\hat{a}_i^\dagger\,\hat{a}_l\right.\\ \left.+\sum_{ik} g_{i0k0}\langle\hat{a}_0\rangle_{st}\,\hat{a}_i^\dagger\,\hat{a}_k+\sum_{jl} g_{0j0l}\langle\hat{a}_0\rangle_{st}\,\hat{a}_j^\dagger\,\hat{a}_l\right)\end{cases}\\
ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\,\hat{a}_0\,\hat{a}_0 &\to i\sum_{kl} g_{00kl}\langle\hat{a}_0^\dagger\rangle_{st}\,\hat{a}_k\,\hat{a}_l\\
ig_{0000}\,\hat{a}_0^\dagger\,\hat{a}_0\,\hat{a}_0 &\to i\frac{1}{2}\left(\sum_{ikl} g_{i0kl}\,\hat{a}_i^\dagger\,\hat{a}_k\,\hat{a}_l+\sum_{jkl} g_{0jkl}\,\hat{a}_j^\dagger\,\hat{a}_k\,\hat{a}_l\right)
\end{aligned}
\tag{3.3.6}
$$

This one to one correspondence illustrates directly that the single mode case is the sum of different modes, in other words, the higher order modes make a contribution to the ground mode, moreover, the coupling between different modes also have the influence on the ground mode.

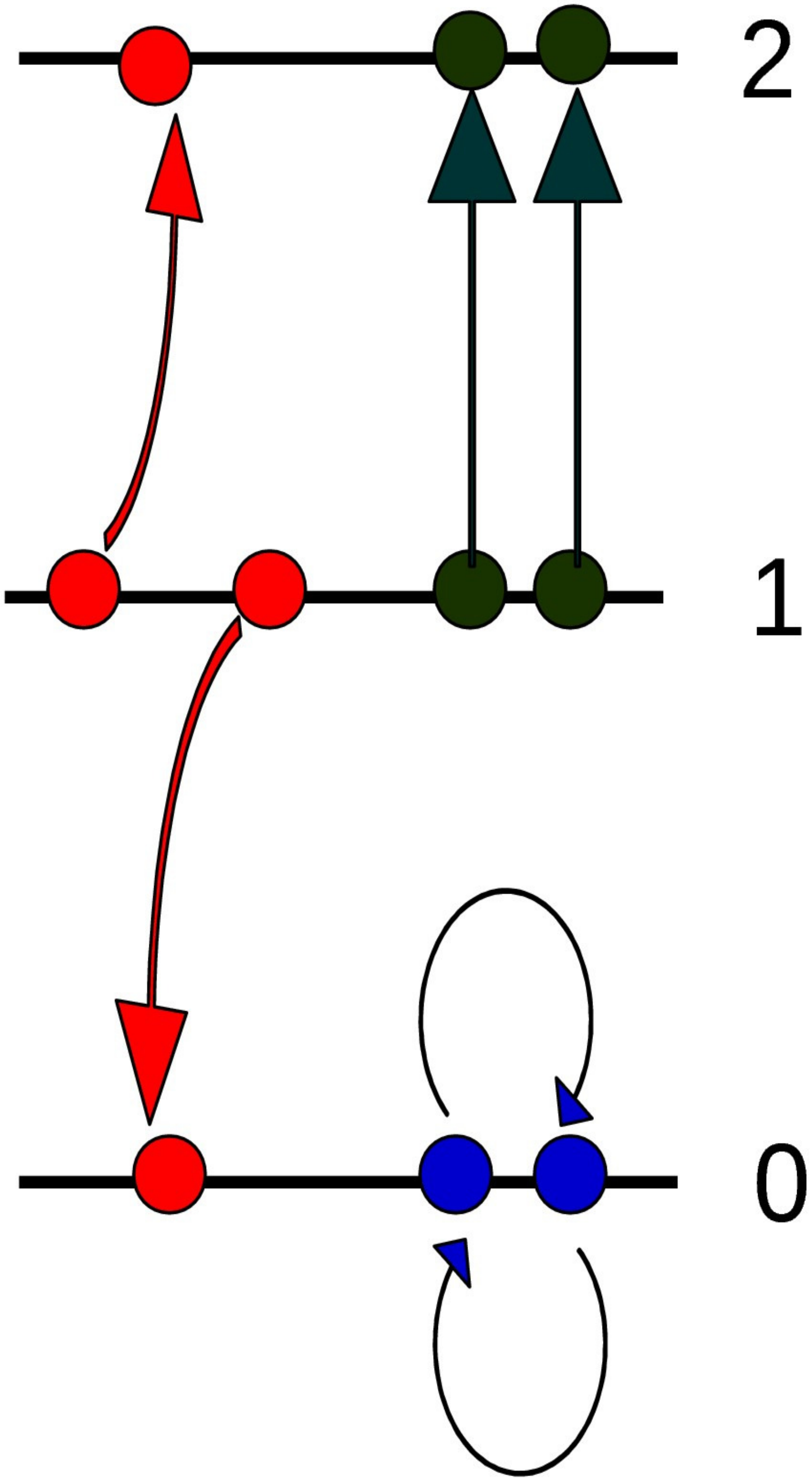

Figure 3.5: Red dashed line: Standard quantum limit. Blue line: The minimal value that the squeezing can reach. Green line: The variance of quadrature operator versus angle . (a) the corresponding intensity is in stable region. (b) the corresponding intensity is in unstable region meanwhile the variance reaches the minimal value. (c) the corresponding intensity is still in unstable region meanwhile the variance reaches the minimal value. (d) the corresponding intensity is in stable region again. Parameters: (a)$\langle \hat{a}_0 \rangle_{st} = \langle \hat{a}_0^{\dagger} \rangle_{st} = 5(b)\langle \hat{a}_0 \rangle_{st} = \langle \hat{a}_0^{\dagger} \rangle_{st} = 5.782(c)\langle \hat{a}_0 \rangle_{st} = \langle \hat{a}_0^{\dagger} \rangle_{st} = 10.2$ (d)$\langle \hat{a}_0 \rangle_{st} = \langle \hat{a}_0^{\dagger} \rangle_{st} = 300$. $g_{0000} = 0.01$meV m^2; $_0 = 0.1$meV

Taking $n = 1$ from Equation (3.3.16)

$$\begin{aligned}
\frac{\partial}{\partial t}\hat a_1 = & -i(\omega_1-\omega_p)\hat a_1 - i\frac{1}{2}\sum_{jkl} g_{1jkl}\Big[\delta_{j0}\delta_{k0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_l + \delta_{k0}\delta_{l0}\langle\hat a_0\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_j^\dagger \\
& + \delta_{j0}\delta_{l0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_k + \delta_{l0}\langle\hat a_0\rangle_{st}\hat a_j^\dagger\hat a_k + \delta_{j0}\langle\hat a_0^\dagger\rangle_{st}\hat a_k\hat a_l + \delta_{k0}\langle\hat a_0\rangle_{st}\hat a_j^\dagger\hat a_l + \hat a_j^\dagger\hat a_k\hat a_l\Big] \\
& - i\frac{1}{2}\sum_{ikl} g_{i1kl}\Big[\delta_{i0}\delta_{k0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_l + \delta_{k0}\delta_{l0}\langle\hat a_0\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_i^\dagger + \delta_{i0}\delta_{l0}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_k \\
& + \delta_{l0}\langle\hat a_0\rangle_{st}\hat a_i^\dagger\hat a_k + \delta_{i0}\langle\hat a_0^\dagger\rangle_{st}\hat a_k\hat a_l + \delta_{k0}\langle\hat a_0\rangle_{st}\hat a_i^\dagger\hat a_l + \hat a_i^\dagger\hat a_k\hat a_l\Big] - \frac{\gamma_1}{2}\hat a_1 + \hat F_1
\end{aligned} \tag{3.3.7}$$

For the sum part $i, j, k, l \in \{0, 1\}$ and $i + j = k + l$ are limited

$$\begin{aligned}
\frac{\partial}{\partial t}\hat a_1 = & -i(\omega_1-\omega_p)\hat a_1 - i\frac{1}{2}g_{1001}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_1 - i\frac{1}{2}g_{1010}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_1 \\
& - i\frac{1}{2}g_{1010}\langle\hat a_0\rangle_{st}\hat a_0^\dagger\hat a_1 - i\frac{1}{2}g_{1001}\langle\hat a_0^\dagger\rangle_{st}\hat a_0\hat a_1 - i\frac{1}{2}g_{1010}\langle\hat a_0^\dagger\rangle_{st}\hat a_1\hat a_0 \\
& - i\frac{1}{2}g_{1001}\langle\hat a_0\rangle_{st}\hat a_0^\dagger\hat a_1 - i\frac{1}{2}g_{1111}\hat a_1^\dagger\hat a_1\hat a_1 - i\frac{1}{2}g_{1001}\hat a_0^\dagger\hat a_0\hat a_1 \\
& - i\frac{1}{2}g_{1010}\hat a_0^\dagger\hat a_1\hat a_0 \\
& - i\frac{1}{2}g_{0101}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_1 - i\frac{1}{2}g_{0110}\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_1 - i\frac{1}{2}g_{0110}\langle\hat a_0\rangle_{st}\hat a_0^\dagger\hat a_1 \\
& - i\frac{1}{2}g_{0101}\langle\hat a_0^\dagger\rangle_{st}\hat a_0\hat a_1 - i\frac{1}{2}g_{0110}\langle\hat a_0^\dagger\rangle_{st}\hat a_1\hat a_0 - i\frac{1}{2}g_{0101}\langle\hat a_0\rangle_{st}\hat a_0^\dagger\hat a_1 \\
& - i\frac{1}{2}g_{0110}\hat a_0^\dagger\hat a_1\hat a_0 - i\frac{1}{2}g_{1111}\hat a_1^\dagger\hat a_1\hat a_1 - i\frac{1}{2}g_{0101}\hat a_0^\dagger\hat a_0\hat a_1 \\
& - \frac{\gamma_1}{2}\hat a_1 + \hat F_1
\end{aligned} \tag{3.3.8}$$

considering the geometric symmetry of nonlinear coupling coefficient $g_{1001} = g_{0101} = g_{0110} = g_{1010}$ resulting from $g_{ijkl} = g\int d\mathbf{r}\,\psi_i^*(\mathbf{r})\psi_j^*(\mathbf{r})\psi_k(\mathbf{r})\psi_l(\mathbf{r})$

$$\begin{aligned}
\frac{\partial}{\partial t}\hat a_1 = & -i\omega_{1p}\hat a_1 - i2g_{1001}\Big(\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st}\hat a_1 + \langle\hat a_0\rangle_{st}\hat a_0^\dagger\hat a_1 + \langle\hat a_0^\dagger\rangle_{st}\hat a_0\hat a_1 \\
& + \hat a_0^\dagger\hat a_0\hat a_1\Big) - ig_{1111}\hat a_1^\dagger\hat a_1\hat a_1 - \frac{\gamma_1}{2}\hat a_1 + \hat F_1
\end{aligned} \tag{3.3.9}$$

where $\omega_{1p} = \omega_1 - \omega_p$ is taken for simplicity

$$\begin{aligned}
\frac{\partial}{\partial t}\hat a_1 = & -\left[i\omega_{1p} + \frac{\gamma_1}{2} + i2g_{1001}\Big(\langle\hat a_0^\dagger\rangle_{st}\langle\hat a_0\rangle_{st} + \langle\hat a_0\rangle_{st}\hat a_0^\dagger + \langle\hat a_0^\dagger\rangle_{st}\hat a_0 + \hat a_0^\dagger\hat a_0\Big)\right]\hat a_1 \\
& - ig_{1111}\hat a_1^\dagger\hat a_1\hat a_1 + \hat F_1
\end{aligned} \tag{3.3.10}$$

$$\begin{pmatrix} i(\omega_{1p}-\omega)+\frac{\gamma_0}{2}+2ig_{1001}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st} & +ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st} \\ -ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st} & -i(\omega_{1p}+\omega)+\frac{\gamma_0}{2}-2ig_{1001}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st} \end{pmatrix}\begin{pmatrix}\tilde{a}_1(\omega)\\ \tilde{a}_1^\dagger(-\omega)\end{pmatrix}=\begin{pmatrix}\tilde{F}_1(\omega)\\ \tilde{F}_1^\dagger(-\omega)\end{pmatrix} \tag{3.3.11}$$

$$\begin{pmatrix}\tilde{a}_1(\omega)\\ \tilde{a}_1^\dagger(-\omega)\end{pmatrix}=\frac{1}{\frac{\gamma_0^2}{4}-i\gamma_0\omega-\omega^2-\langle\hat{a}_0^\dagger\rangle_{st}^2\langle\hat{a}_0\rangle_{st}^2g_{0000}^2+4\langle\hat{a}_0^\dagger\rangle_{st}^2\langle\hat{a}_0\rangle_{st}^2g_{1001}^2+4\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}g_{1001}\omega_{1p}+\omega_{1p}^2}$$
$$\begin{pmatrix}\frac{\gamma_0}{2}-2ig_{1001}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}-i(\omega+\omega_{1p}) & -ig_{0000}\langle\hat{a}_0\rangle_{st}^2 \\ ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}^2 & \frac{\gamma_0}{2}+2ig_{1001}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}+i(-\omega+\omega_{1p})\end{pmatrix}\begin{pmatrix}\tilde{F}_1(\omega)\\ \tilde{F}_1^\dagger(-\omega)\end{pmatrix} \tag{3.3.12}$$

thus we can get

$$\tilde{a}_1(\omega)\,\tilde{a}_1^\dagger(-\omega)=\frac{1}{\left(\frac{\gamma_0^2}{4}-i\gamma_0\omega-\omega^2-\langle\hat{a}_0^\dagger\rangle_{st}^2\langle\hat{a}_0\rangle_{st}^2g_{0000}^2+4\langle\hat{a}_0^\dagger\rangle_{st}^2\langle\hat{a}_0\rangle_{st}^2g_{1001}^2+4\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}g_{1001}\omega_{1p}+\omega_{1p}^2\right)^2}$$
$$\frac{1}{4}i\left(-2i\omega+2i\langle\hat{a}_0^\dagger\rangle_{st}\left(\langle\hat{a}_0^\dagger\rangle_{st}g_{0000}+2\langle\hat{a}_0\rangle_{st}g_{1001}\right)+2i\omega_{1p}\right)\left(i\gamma+2\omega+2\langle\hat{a}_0\rangle_{st}^2g_{0000}+4\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}g_{1001}+2\omega_{1p}\right)$$
$$\tilde{F}_1(\omega)\,\tilde{F}_1^\dagger(-\omega) \tag{3.3.13}$$

Following the same condition, we obtain the dynamics for $\hat{a}_0$

$$\frac{\partial}{\partial t}\hat{a}_0=-\left(i\omega_{0p}+\frac{\gamma_0}{2}+i2g_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}+i2g_{1001}\langle\hat{a}_1^\dagger\hat{a}_1\rangle\right)\hat{a}_0-ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st}\hat{a}_0^\dagger$$
$$-ig_{0000}\left(\langle\hat{a}_0^\dagger\rangle_{st}\hat{a}_0\hat{a}_0+\hat{a}_0^\dagger\hat{a}_0\hat{a}_0+2\langle\hat{a}_0\rangle_{st}\hat{a}_0^\dagger\hat{a}_0\right)-i2g_{1001}\langle\hat{a}_0\rangle_{st}\hat{a}_1^\dagger\hat{a}_1+\hat{F}_0 \tag{3.3.14}$$

$$\begin{pmatrix} i(\omega_{1p}-\omega)+\frac{\gamma_0}{2}+2ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0\rangle_{st}+2ig_{1001}\langle\hat{a}_1^\dagger\hat{a}_1\rangle & +ig_{0000}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0\rangle_{st} \\ -ig_{0000}\langle\hat{a}_0^\dagger\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st} & -i(\omega_{1p}+\omega)+\frac{\gamma_0}{2}-2ig_{1001}\langle\hat{a}_0\rangle_{st}\langle\hat{a}_0^\dagger\rangle_{st}-2ig_{100\ldots} \end{pmatrix}\begin{pmatrix}\tilde{a}_0(\omega)\\ \tilde{a}_0^\dagger(-\omega)\end{pmatrix}=\begin{pmatrix}\tilde{F}_0(\omega)\\ \tilde{F}_0^\dagger(-\omega)\end{pmatrix} \tag{3.3.15}$$

With considering only the influence from the first state, the

$$i\,!_{0p} \quad \frac{\theta}{2} \quad 2ig_{0000}h\hat{a}_0^y i_{st}h\hat{a}_0 i_{st} \quad i2g_{1001}\ \hat{a}_1^y\ \hat{a}_1 \qquad ig_{0000}h\hat{a}_0 i_{st}h\hat{a}_0 i_{st} \quad !$$

$$ig_{0000}h\hat{a}_0^y i_{st}h\hat{a}_0^y i_{st} \qquad i\,!_{0p} \quad \frac{\theta}{2} + 2ig_{0000}h\hat{a}_0 i_{st}h\hat{a}_0^y i_{st} + i2g_{1001}\ \hat{a}_1^y\ \hat{a}_1 \tag{3.3.16}$$

Appendix A

The Matlab code for Figure 3.1, Figure 3.2 and Figure 3.3

```
%Size of the axes
ax = gca;
ax.FontSize = 15;

% Generate values from our functions

n=(0:0.01:140);
t=140;
n1=(0:0.01:33.4585);
n2=(99.86:0.01:140);

%Given the constant

gamma=0.1;
g=0.01;
omegaop=-1;

% Input the given functions

Sqrt1 = sqrt(omegaop.^2+4*omegaop*g*n+3*g.^2*n.^2);
Sqrt2 = sqrt(omegaop.^2+4*omegaop*g*n);
Lambda1 = -gamma/2+sqrt(-(omegaop+2*g*n).^2+(g*n).^2);
Lambda2 = -gamma/2-sqrt(-(omegaop+2*g*n).^2+(g*n).^2);
Intensity = ((gamma.^2)/4+omegaop.^2)*n+2*g*omegaop*(n.^2)+(g.^2)*(n.^3);
PhotoNumberMarkPointX= 33.4585:0.01:99.9;
IntensityMarkPointY= IntensityCalculator(PhotoNumberMarkPointX);
EigenvalueMarkPointY= EigenValueCalculator(PhotoNumberMarkPointX);
```

```
StrightDashedLine= StrightLine(n);
hold all
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Eigenvalue

p1=plot(n1, EigenValueCalculator(n1),'r','linewidth',2);
p2=plot(n2, EigenValueCalculator(n2),'r','linewidth',2);
p3=plot(n, Lambda2,'r','linewidth',2);
p4=plot(PhotoNumberMarkPointX,
    EigenvalueMarkPointY,'Color','blue','linewidth',2);
p5=plot(n, StrightDashedLine,'--','Color','Black','linewidth',1);
xlabel('Photo number $\mathrm{n_0}$','linewidth',2, 'Interpreter', 'latex') %
    x-axis label
ylabel('Eigenvalue $\lambda_{\pm}$','linewidth',2, 'Interpreter', 'latex') %
    y-axis label
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Intensity

p6=plot(IntensityCalculator(PhotoNumberMarkPointX),
    PhotoNumberMarkPointX,'b','linewidth',2);
p7=plot(IntensityCalculator(n1), n1,'Color','red','linewidth', 2);
p8=plot(IntensityCalculator(n2), n2,'Color','red','linewidth', 2);
xlabel('Intensity $\mathrm{I_0}$', 'linewidth',2,'Interpreter','latex') %
    x-axis label
ylabel('Photo number $\mathrm{n_0}$','linewidth',2, 'Interpreter', 'latex') %
    y-axis label
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Eigenvalue And Intensity Together

p1=plot(EigenValueCalculator(n1),n1,'r','linewidth',2);
p2=plot(EigenValueCalculator(n2),n2,'r','linewidth',2);
p3=plot(Lambda2,n,'r','linewidth',2);
p4=plot(Lambda1,n,'r','linewidth',2);
p5=plot(EigenvalueMarkPointY,PhotoNumberMarkPointX
    ,'Color','blue','linewidth',2);
p6=plot(IntensityCalculator(PhotoNumberMarkPointX),
    PhotoNumberMarkPointX,'b','linewidth',2);
p7=plot(IntensityCalculator(n1), n1,'Color','red','linewidth', 2);
p8=plot(IntensityCalculator(n2), n2,'Color','red','linewidth', 2);
xlabel('Intensity $\mathrm{I_0}$', 'linewidth',2,'Interpreter','latex') %
    x-axis label
ylabel('Photo number $\mathrm{n_0}$','linewidth',2, 'Interpreter', 'latex') %
    y-axis label
```

```
% create a new pair of axes inside current figure

axes('position',[.55 .55 .48 .3])
box on % put box around new pair of axes
hold all
p9=plot(EigenValueCalculator(95:0.01:99.86),95:0.01:99.86,'b','linewidth',2);
p10=plot(EigenValueCalculator(99.86:0.01:105),99.86:0.01:105,'r','linewidth',2);
p11=plot(IntensityCalculator(95:0.01:99.86), 95:0.01:99.86,'b','linewidth',2);
p12=plot(IntensityCalculator(99.86:0.01:105),
    99.86:0.01:105,'r','linewidth',2);
axis tight
%grid off
%pbaspect([10 10 10])
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
grid off
pbaspect([2 1 1])
```

Appendix B

The Mathematica code for Figure 3.5

```
(* the package "SciDraw" is used *)

["SciDraw`"];
define example graphics *)
mula = (E^(-2 I \[Theta]) (-E^(4 I \[Theta])
       g oadost^2 (4 g oadost oaost - I \[Gamma]o +
       2 Subscript[\[Omega], op]) -
    g oaost^2 (4 g oadost oaost + I \[Gamma]o +
       2 Subscript[\[Omega], op]) +
    E^(2 I \[Theta]) (16 g^2 oadost^2 oaost^2 + \[Gamma]o^2 +
       4 Subscript[\[Omega],
        op] (4 g oadost oaost + Subscript[\[Omega],
          op]))))/(12 g^2 oadost^2 oaost^2 + \[Gamma]o^2 +
  4 Subscript[\[Omega],
   op] (4 g oadost oaost + Subscript[\[Omega], op]));
ure1 =
lot[{1, 0.5,
 formula /. {g -> 0.01, \[Gamma]o -> 0.1,
   Subscript[\[Omega], op] -> -1, oadost -> 5,
   oaost -> 5}}, {\[Theta], 0, 2 \[Pi]}, PlotRange -> {0.3, 2.1},
PlotStyle -> {Directive[RGBColor[0.76, 0., 0.04], Dashed], Blue,
  DarkGreen}];
ure2 =
lot[{1, 0.5,
 formula /. {g -> 0.01, \[Gamma]o -> 0.1,
   Subscript[\[Omega], op] -> -1, oadost -> 5.782,
   oaost -> 5.782}}, {\[Theta], 0, 2 \[Pi]},
```

```
PlotRange -> {0.3, 2.1},
PlotStyle -> {Directive[RGBColor[0.76, 0., 0.04], Dashed], Blue,
  DarkGreen}];
ure3 =
lot[{1, 0.5,
 formula /. {g -> 0.01, \[Gamma]o -> 0.1,
   Subscript[\[Omega], op] -> -1, oadost -> 10.02,
   oaost -> 10.02}}, {\[Theta], 0, 2 \[Pi]},
PlotRange -> {0.3, 2.1},
PlotStyle -> {Directive[RGBColor[0.76, 0., 0.04], Dashed], Blue,
  DarkGreen}];
ure4 =
lot[{1, 0.5,
 formula /. {g -> 0.01, \[Gamma]o -> 0.1,
   Subscript[\[Omega], op] -> -1, oadost -> 300,
   oaost -> 300}}, {\[Theta], 0, 2 \[Pi]}, PlotRange -> {0.3, 2.1},
 PlotStyle -> {Directive[RGBColor[0.76, 0., 0.04], Dashed], Blue,
  DarkGreen}];
create multipanel figure from these graphics *)

ure[
ltipanel[


(* set panel label properties *)

SetOptions[FigLabel, FontSize -> 12, TextBackground -> LightGray,
 TextMargin -> 2, TextRoundingRadius -> 2];

(* panel 1 *)
FigurePanel[
 {
  FigGraphics[figure1];
  FigLabel -> False;
  },
 {1, 1},
 XPlotRange -> {0, 2 \[Pi]}, YPlotRange -> {0.3, 2.1},
 PanelLetterTextBackground -> Automatic, (*
 hide plot behind panel letter *)
 ExtendRange -> Automatic (*
 allow some space around the edges of the plot *)
 ];

(* panel 2 *)
```

```
FigurePanel[
 {
  FigGraphics[figure2];
  FigLabel -> False;
  },
 {1, 2},
 XPlotRange -> {0, 2 \[Pi]}, YPlotRange -> {0.3, 2.1}
 ];

(* panel 3 *)
FigurePanel[
 {
  FigGraphics[figure3];
  FigLabel -> False;
  },
 {2, 1},
 XPlotRange -> {0, 2 \[Pi]}, YPlotRange -> {0.3, 2.1}
 ];

(* panel 4 *)
FigurePanel[
 {
  FigGraphics[figure4];
  FigLabel -> False;
  },
 {2, 2},
 XPlotRange -> {0, 2 \[Pi]}, YPlotRange -> {0.3, 2.1}
 ];

},
imensions -> {2, 2},
PanelSizes -> {2, 2}, XPanelGaps -> 0.04,
PanelSizes -> {1, 1}, YPanelGaps -> 0.06,
FrameLabel -> {"\[Theta]", "\[Theta]"},
Ticks -> {LinTicks[0, 3 Pi/2, Pi/2, 4,
  TickLabelFunction -> (Rationalize[#/Pi]*Pi &)],
 LinTicks[0, 2*Pi, Pi/2, 4,
  TickLabelFunction -> (Rationalize[#/Pi]*Pi &)]},
FrameLabel ->
textit["(\[CapitalDelta]X\!\(\*SuperscriptBox[\(()\), \(2\)]\)"]
```

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