Study of entanglement in Hawking process arising in a polariton based acoustic black hole

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Abstract

Analogue gravity experiment enables us to study the Hawking process occurring in astrophysical black hole and quantum nature of it. There have been multiple ideas triggered to study the quantum effects of Hawking radiation that emerges at the black hole horizon. We will focus on the exciton-polariton condensate-based system among all the experimental realization of acoustic analogue black hole. To investigate the quantum entanglement in analogue Hawking emission from background phonons at the horizon, we first studied the dynamics of the system governed by an effective Gross-Pitaevskii equation and then use the density-density correlations between emitted pairs of field modes, which are arising due to the quantum fluctuation on both sides of the sonic horizon. To identify the formation of entanglement, we use the covariance matrix formulation, and to analyze the non-separability of quantum modes, positive partial transpose criteria have been adopted. The final aim of this project is to compute the degree of formation of entanglement by using the quantities which are accessible with experimental data.

Contents

Chapter 1

General Introduction

In 1974, prof Hawking predicted that astrophysical black holes emit faint radiation, and this was when gravitational physics was explained with the quantum picture [1]. Before this, in a study of black hole formation and evolution, it was assumed that the energy density of particles created due to the gravitational field is small compare to the other parameter that governs the physics. Still, neglecting quantum effects was not a correct way to do actual physics. The First article by prof Hawking presents that the black holes radiate thermal fields, and resultant it was predicted that due to this process, the black hole would lose a noticeable amount of mass; thus, black holes have a finite lifetime. There are multiple supporting theoretical arguments and I will choose to not comment much on that. Due to lack of experimental facility, we can not study the so called Hawking process. In 1981, Unruh suggested to use hydrodynamic analogous gravitational black holes to study their quantum properties under a laboratory conditions [2]. Since then, an analogue black hole which comes under a broad topic of analogue gravity enables us to study an astrophysical black hole using the table top experiment. In the first proposal of an analogue experiment, prof Unruh considered a demonstration, where sound waves are propagating in fluid flow. Under this idea, fluid speed happens to be supersonic in some regime and subsonic in other parts of the setup. The boundary between these two regions of the fluid flow is treated as an acoustic horizon. By linearizing the hydrodynamic equations, prof Unruh showed that the motion of sound waves is equivalent to the propagating scalar field in curved space-time. by extending the same analogy, if we quantify the sound field, governing physics should be similar to the quantum field propagating in curved space-time.
To realize this acoustic black hole, till now, multiple abstractions have been presented, such as Bose-Einstein condensate, super-fluid helium, ultra-cold Fermions, light in a nonlinear liquid, exciton-polariton condensate, and surface waves in water. The Quantum optics team at LKB, Paris, lead by Prof Alberto Bramati, is working on the Exciton-Polariton condensate-based system. In polariton-based BEC, non-linearity arises from polariton-polariton interaction, which affects the

properties of the fluid as a whole. In the context of BEC, in 1947, Bogoiubov was the first who attempted to solve the problem of weakly interacting Bose gas and suggested a first principle of mean-field theory of super-fluidity. This theory has been tremendously crucial to explain the emergence of sounds like elementary excitons from quantum fluctuations in the condensate [3]. This Bogoliubov theory plays significant role in the domain of analogue gravity and opens a path to study the quantum effects of the Hawking process.
This report is organised in the following way. In Chapter 2 of the report, necessary basic concepts and their mathematical representation have been summarized. It concludes with sonic horizon configuration and a schematic sketch of 1-D experiment. Chapter 3 is devoted for numerical schemes to solve the dynamics of the system and a few preliminary results related to the correlation, emerging in the Hawking process. The parameters to get the correlations are defined as per the LKB's experiment specifications. In Chapter 4, we study quantum effects of the Hawking process specifically bipartite entanglement and different measurement schemes. The last Chapter 5 of this report is assigned for the conclusions derived from this internship, and future expected results.

Chapter 2

Realization of acoustic analogous polariton black hole

In quest of a promising system to study quantum hydrodynamics effects specifically analogue black hole and Hawking process, quantum fluid of light emerges as a potential system. Among different optical platforms, the installment of photons in semiconductor micro-cavity in a strong coupling regime known as cavity polaritons has led to all the expected properties of quantum fluid needed to study the Hawking process. In 2015, a team of researchers led by Carusotto and Bloch [4] suggested the possibility of a black-hole configuration in flowing polariton condensate first time. At LKB, installed system to study the Hawking process is similar to the Carusotto's experiment. Further, technical discussion in more detail have been organized in this chapter. In order to realize an analogue acoustic polariton-based black hole, this chapter talks about the physical system, effective photonic interaction inside the micro-cavity, and the coupling between excitons and photons, which led to the formation of polariton fluid governed by the Bose-Einstein statistics. To derive the picture of Hawking process, Bogoliubov approximation and transformation have been introduced, which yields the stationary solution of the corresponding Gross-Pitaevskii Equation. At last, the chapter is concluded with a brief discussion on acoustic horizon configuration and different modes of the Hawking process arising at the sonic Horizon.

2.1 System: Light Matter Interaction and Polariton Formation

2.1.1 Micro-cavity and effective photonic interaction

The system is a semiconductor micro cavity, consists of InGaAs quantum wells installed between two Bragg mirrors. This architecture forms a thin cavity of high finesse. In many photon system, collision between photon lead to effective fluid-like behaviors. The photon-photon interaction have been broadly explained via virtual excitation of electron and positron pairs. In the proceeding section of this report, excitons are meant by the pairs of electron and hole. The picture of photon-photon interaction is hardly useful to study any physical phenomena. On the other hand interaction mediated by nonlinear medium emerges as a potential system. Many different configuration have been studied in the past decade which comes under strong light matter interaction physics. By intuition of the same strong light matter coupling, when photons interact with the excitons, it creates a new mixed quantum fluid of quasi-particles called polariton [5][6]. To maintain the stability inside cavity, it is important to give a finite effective mass to a photon. This comes via the system architecture itself. A spatial confinement of the photon by using Bragg mirrors gives the effective mass to a photon. In a planer configuration of Bragg mirrors with a dielectric medium of refractive index n_0 and thickness l_z, the photon motion along the z axis is quantize as $q_z = \frac{\pi N}{l_z}$. where N is a positive integer. Inside the cavity, for each longitudinal mode, the frequency dispersion depends on the in plane wave vector k, which is expressed as follows,

$$\omega_c(k) = \frac{c}{n_0}\sqrt{q_z^2 + k^2} \simeq \omega_c^0 + \frac{\hbar k^2}{2m_c} \tag{2.1}$$

where the effective mass m_c of the photon and cutoff frequency $\omega_c^0 = cq_z/n_0$ are related by the relativistic expression,

$$m_c = \frac{\hbar n_0 q_z}{c} \tag{2.2}$$

In presence of the resonant electronic excitation coupled with cavity mode, we can achieve polaritonic character in the elementary excitations of the cavity. These polariton fluid shows a peculiar dispersion relation, which reflects their hybrid light matter interaction properties [5]. A detailed mathematical construction of polariton has been presented in the forgoing section.

2.1.2 Quantum well excitons coupled to the field modes

Exciton in quantum wells is pair of electron and hole, bound via coulomb interaction. their energy for a given wave vector is read as follows,

$$E_x(k) = E_{bg} + E_Q - \frac{\hbar^2}{2m_r a_B^2}\frac{1}{(n-1/2)^2} + \frac{\hbar^2 k_{//}^2}{2M} \tag{2.3}$$

Where first term is band gap energy of QW semiconducting material. Second term denotes the quantization energy due to confinement of excitons. Third term summarize the binding energy resulted from Coulomb interaction in quantum well. m_r and a_B denote the reduced mass of excitons and Bohr radius respectively. Fourth term is due to incident laser beam on quantum well. If incidence angle is θ and frequency ω, the *inplane* wave vector $k_{||}$ is given by $(\omega/c)\sin\theta$. M is the total mass of the exciton.

Energy of the trapped photons inside micro cavity is expressed as follows,

$$E_c(k_{xy}) = \frac{\hbar c}{n_c}k_z + \frac{1}{2}\frac{\hbar c}{n_c}\frac{k_{xy}^2}{k_z}. \tag{2.4}$$

Strong coupling: Energy exchange between excitons and photons inside a high quality micro cavity leads to the formation of two new eigen states which are coherent superposition of two initial state. Energy separation between these two new eigen states are related with the coupling energy between cavity photon and exciton [5].

Initially without coupling, the Hamiltonian of the system is as follows,

$$\hat{H}_0 = \sum_k E_x(k)\hat{b}_k^\dagger \hat{b}_k + \sum_k E_c(k)\hat{c}_k^\dagger \hat{c}_k, \tag{2.5}$$

Where $\hat{b}^\dagger, \hat{b}$ and $\hat{c}^\dagger, \hat{c}$ are the creation and annihilation operators of excitons and photons respectively. After coupling between constituents, the system Hamiltonian is expressed as follows,

$$H_{xc} = \sum_k \hbar\Omega\left(\hat{b}_k^\dagger \hat{c}_k + \hat{c}_k^\dagger \hat{b}_k\right) \tag{2.6}$$

Where, $\hbar\Omega$ reads the Interaction strength. The interaction matrices can be written with 2×2 dimension, $M = \begin{bmatrix} E_x(k) & \hbar\Omega \\ \hbar\Omega & E_c(k) \end{bmatrix}$

To get the eigenvalue of interaction matrices, we can use general formulation, that gives

$$E_{U,L}(k) = \frac{E_x(k) + E_c(k)}{2} \pm \frac{1}{2}\sqrt{(E_c(k) - E_x(k))^2 + 4\hbar^2\Omega^2} \tag{2.7}$$

Where E_U and E_L co-responds to the upper and lower polariton energy branch. As we are constructing the basic building blocks of an experiment and theoretical description of it, it is important to say that the expression of interaction Hamiltonian is based on the so-called rotating wave approximation [5]. According to this approximation, anti-resonant term proportional to $\hat{b}_k\hat{c}_k$ are neglected. This theory works very well as long as $\Omega_R << \omega_c, \omega_e$, generally we are able to achieve this regime in micro cavity experiment. For rest of the report, our main focus will be on lower polariton branch only and dynamics will be expressed in terms of lower polariton wave function. In the next section, effective model Hamiltonian has been discussed with respect to the lower polariton branch.

2.1.3 Exciton polaritons and interaction

Polaritons are the integer spin quasi particles governed by the Bose-Einstein statistics. As we know Bosons have infinite compressibiltiy, comes from the fact that Bosons can be accommodated in a single state regardless to Pauli Exclusion principle. Therefore it is not a big deal to experience the gas's property change due to interaction between Bosons. Even if they are in ground state, they can have fluctuation induced by interaction. As the ground state has zero energy in absence of the interaction, and this pose a problem which can not be solved with traditional perturbation theory. In 1947, Bogoliubov proposed a new perturbation techniques and provided a new approach to solve the problem of fluctuation in BEC like fluid [3].

2.2 Dynamics of the system

2.2.1 Bogoliubov picture of quantum fluctuation

Let us first construct the order parameter and one body density matrices for BEC and then we will extend the same ideology for polariton fluid. If the single-particle wave function and its eigenvalue is given by φ_i and n_i respectively, it allows us to write the density matrix in a very useful diagonalized form, which is expressed as follows [7],

$$n^{(1)}(r, r') = \Sigma_i n_i \varphi_i^*(r)\varphi_i(r') \tag{2.8}$$

Where the eigenvalues n_i are defined under the second quantisation which provide the single-particle occupation number related to the single-particle states φ_i.
The observation of BEC can be made when one of the single-particle states is occupied in a macroscopic way i.e. when $n_{i=0} = N_0$ is a number of order N (total number of particle), while the other single-particle states have a microscopic occupation of order 1. In this picture, (2.8) can be expressed as follows,

$$n^{(1)}(r, r') = N_0 \varphi_0^*(r)\varphi_0(r') + \Sigma_{i>0} n_i \varphi_i^*(r)\varphi_i(r') \tag{2.9}$$

The single-particle wave function φ_i can be used to express the field operator $\hat{\Psi}(r)$,

$$\hat{\Psi}(r) = \Sigma_i \varphi_i \hat{a}_i \tag{2.10}$$

where, $\hat{a}_i(\hat{a}_i^\dagger)$ are the annihilation (creation) operators of a particle in the state φ_i and obey the canonical commutation relations given by,

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \tag{2.11}$$

In order to study the fluctuation in BEC, it is useful to write the field operator given by (2.10), in terms of condensate with $i = 0$ and other components,

$$\hat{\Psi}(r) = \varphi_0(r)\hat{a}_0 + \Sigma_{i>0} \varphi_i \hat{a}_i \tag{2.12}$$

Till now all the formulations are applicable for polariton condensate as well. This is perfect time to introduce the Bogoliubov approximation. For the macroscopic phenomena associated with BEC, Bogoliubov approximation replaces the particle creation and annihilation operator with $\sqrt{N_0}$, which is equivalent to the ignoring the non-commutivity of these operators. Here, $N_0 = \langle \hat{a}_0^\dagger \hat{a}_0 \rangle \gg 1$. This approximation allows us to treat the first macroscopic term $\varphi_0 \hat{a}_0$ in (2.12) as a classical field so that equation 2.12 can be re-expressed as,

$$\hat{\Psi}(r) = \Psi_0(r) + \delta\hat{\Psi}(r) \tag{2.13}$$

where the first term in right hand side (2.13) carries the Bogoliubov approximation and defined as $\Psi_0 = \sqrt{N_0}\varphi_0$ and second term is written with $\delta\hat{\Psi} = \Sigma_{i \neq 0} \varphi_i a_i$, which contains the information about quantum fluctuation of BEC fluid affected by the particle interaction and local temperature [7]. In the next section, we will extend the same intuition to calculate the two-body density and see how quantum fluctuation changes.

2.2.2 Two-body density correlation

The ideal Bose gas makes the researcher's life easy and one body density matrix determines the behavior of two body density, which is expressed as,

$$n^{(2)}(r, r') = \langle \hat{\Psi}^{\dagger}(r)\hat{\Psi}^{\dagger}(r')\hat{\Psi}(r')\hat{\Psi}(r)\rangle \tag{2.14}$$

The above equation can be also expressed in terms of the single-particle operators by expanding the field operator using Bogoliubov definition. For polariton fluid equation (2.13) can be rewritten as,

$$\hat{\Psi}(r) = \Psi_{LP}(r) + \delta\hat{\Psi}(r) \tag{2.15}$$

The condensate probability distribution function: it's number fluctuation defined by the normalized second order correlation function $g^{(2)}(r, t)$ introduced by Glauber in 1963 [8]. Here, a brief formulation is presented and in the numerical section, a broad view has been presented under Wigner theory. the correlation function $g^{(2)}(r, t)$ is defined as,

$$g^{(2)}(r, r') = \frac{\langle \hat{\Psi}^{\dagger}(r)\hat{\Psi}^{\dagger}(r')\hat{\Psi}(r')\hat{\Psi}(r)\rangle}{\langle \hat{\Psi}^{\dagger}(r)\hat{\Psi}(r)\rangle\langle \hat{\Psi}^{\dagger}(r')\hat{\Psi}(r')\rangle} \tag{2.16}$$

More qualitatively, $g^{(2)}(r)$ tells us the degree of spatial correlation between two points at the same time.

2.2.3 The Gross-Pitaevskii Equation

To study the interacting nonuniform condensate one must generalize the Bogoliubov theory introduced in the previous section. In very low temperature condition, it is applicable to generalize the equation (2.15) and replace the field operator $\hat{\Psi}(r)$ with wave function of condensate $\Psi_{LP}(r)$ which is also corresponds to the classical field. In fact this replacement is the analogous to the transition of quantum electrodynamics to the classical picture of electromagnetism. As we know in the actual experimental setup, we have very large number of condensate in the same quantum state, so above made approximation is justified to give the expected outcomes.

To study the dynamics of this field, we directly adopt the Heisenberg representation of the field operator under driven dissipative condition, which was derived independently by Gross (1961) and Pitaevskii (1961) and it is the main theoretical tool to study the nonuniform dilute Bose gases at low temperature [9]. It is expressed as,

$$i\partial_t \Psi_{LP}(r,t) = \left[\omega_{LP}^0 - \frac{\hbar}{2m_{LP}}\nabla^2 + V_{LP}(r)\right]\Psi_{LP}(r,t) + g_{LP}|\Psi_{LP}(r,t)|^2\Psi_{LP}(r,t) - \frac{i\gamma_{LP}}{2}\Psi_{LP}(r,t) + i\eta_{LP}E(r,t) \tag{2.17}$$

Where ω_{LP}^0 is frequency of the lower polariton (LP) branch in ground state, m_{LP} is mass of the polariton, V_{LP} is the external potential felt by polaritons inside a cavity, g_{LP} is polariton-polariton interaction constant which gives the strength order of non linear interactions, γ_{LP} is the proportionality coefficient for loss rate, η_{LP} quantifies the coupling of polariton to the incident light and E is amplitude of the coherent pump [5].
Before moving further, it is better to summarize the conditions under which, equation (2.17) is applicable. The first condition is to have large number of condensate. Second, temperature should be low enough to replace the field operator with classical field. With all these approximation, it is a difficult task to have the exact results with quantum and thermal fluctuation. To solve this problem, there have been number of proposal, which we will discuss in the next chapter dedicated for numerical analysis.

2.2.4 Bogoliubov dispersion

Bogoliubov dispersion of collective excitons is obtained by linearizing the GPE with the steady state solution. For our purpose, we will only focus on a case where coherent pump intensity is set to at the turning point of the upper branch of the bistability loop, which comes from the fact of the non-linearity of medium. [10]. Lineraizing the GPE around the steady solution leads to a pair of equation given as,

$$i\partial_t \begin{pmatrix} \delta\Psi_{LP}(r) \\ \delta\Psi_{LP}^*(r) \end{pmatrix} = \mathcal{L}_{Bog} \begin{pmatrix} \delta\Psi_{LP}(r) \\ \delta\Psi_{LP}^*(r) \end{pmatrix} \tag{2.18}$$

with the Bogoliubov operator $\mathcal{L}_{Bog}$. To get the idea of Bogoliubov dispersion, only one case of a spatially homogeneous system with $V_{LP}(r) = 0$ and a coherent pump with $k_{inc} = 0$ is expressed. Under these assumption, the stationary state or steady state has the form $\Psi_{LP}(r,t) = \sqrt{n_{LP}} exp(-i\omega_{inc}t)$ where $\mathcal{L}_{Bog}$ expressed as,

$$\mathcal{L}_{Bog} = \begin{pmatrix} \omega_{LP}(k) + 2g_{LP}n_{LP} - \frac{i\gamma_{LP}}{2} - \omega_{inc} & g_{LP}n_{LP} \\ -g_{LP}n_{LP} & -\omega_{LP}(k) - 2g_{LP}n_{LP} - \frac{i\gamma_{LP}}{2} + \omega_{inc} \end{pmatrix} \quad (2.19)$$

and its eigenvalues describe the so-called Bogoliubov dispersion of excitations expressed as,

$$\omega_{\text{Bog}}(k) = \pm\sqrt{(\frac{\hbar k^2}{2m_{LP}} + \omega^0_{LP} + 2g_{LP}n_{LP} - \omega_{inc})^2 - (g_{LP}n_{LP})^2} - i\frac{\gamma_{LP}}{2} \quad (2.20)$$

Here $\pm$ sign refers to the positive and negative Bogoliubov branch. This matrix for coherently pumped polariton condensate flowing with the speed $v = \hbar k_{inc}/m_{LP}$ was first explained by Carusotto and Ciuti in 2005 [11]. Experimentally, it was confirmed in 2009 by Amo Lefrere et al [4]. The system behaviour is being widely explained using the ratio between the wavelength and the shortest distance under which the wave function observes some changes. This distance is called as healing length, defined as $\xi = \sqrt{\hbar m_{LP} g_{LP} n_{LP}}$.

2.2.5 An acoustic horizon's configuration

After reviewing all the physical system specifications and technical tools to study the dynamics of polariton condensate fluid, now we can discuss the specific configuration of an acoustic horizon and in the next chapter we will develop a few necessary building blocks to investigate the system numerically.

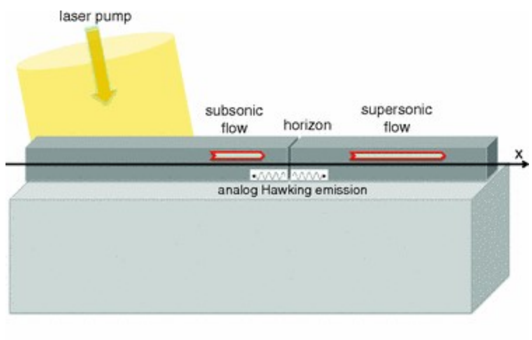

Figure 2.1: Schematic sketch of 1-D polaritons horizon

Figure (2.1) is self explained, which is about schematic diagram of the experimental setup. While (2.2) shows the spatial plots of the condensate flow velocity

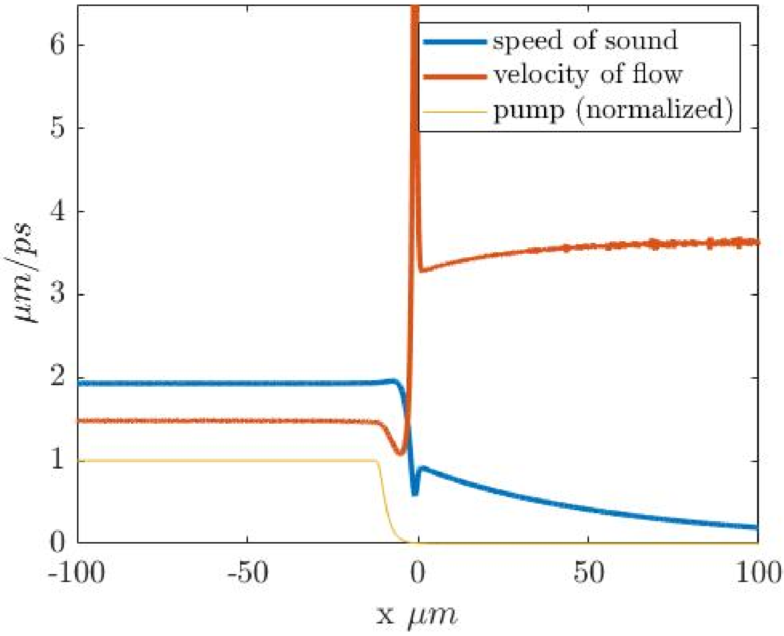

Figure 2.2: Spatial plot of the polariton flow velocity

and the local speed of sound along 1-D system. It is obtained by solving the GPE (2.17) in the stationary state with specific experimental parameters. Numerical scheme to solve the GPE is explained in the next chapter. With respect to figure (2.1), the sonic horizon is located where the fluid speed (represented by red line) exceeds the speed of sound. In the numerical analysis it is defined at $520\mu m$.

Chapter 3

Numerical scheme and simulation's results

3.1 Truncated Wigner approximation (TWA)

The main principle behind Truncated Wigner approximation is to first generate an ensemble of classical fields $\Psi(r)$, which illustrate Wigner quasi-distribution function of initial gas density operator at thermal equilibrium. And in the final stage, evolve each classical field with GPE. There are other approaches such as time dependent Bogoliubov, but it neglects the interaction between condensates and non-condensate particles.

3.1.1 Basic introduction and notations

In case of the steady polariton condensate fluid at thermal equilibrium, the first step is to get the Wigner quasi-distribution function associated with N-body density operator (general case), which is also a function of the classical field $\Psi(r)$. Once we have Wigner function, it can be used to sample it numerically to calculate the mean of observable and probability distribution via correlation function. The regime of the validity of TWA holds the Bogoliubov approximation explained in preceding chapter. In the conventional Bogoliubov approach, we diagonalize the Bogoliubov matrix which gives a bad time (It takes very long time to perform the calculations) in case of 2D or 3D system [12]. TWA adopt the stochastic formulation which gives the access to single realizations and the probability distribution of polariton condensate.
Let us first introduce the Wigner representation for a simple quantity of the average one-body density of condensate, which is expressed as,

$$\langle\hat{\Psi}^\dagger(r)\hat{\Psi}(r)\rangle = \langle\psi^*(r)\psi(r)\rangle_W - \frac{1}{2}\langle[\hat{\Psi}(r), \hat{\Psi}^\dagger(r)]\rangle \tag{3.1}$$

where $\langle ... \rangle_W$ express the mean over the Wigner quasi-distribution function. For the continuous field problem, $[\hat{\Psi}(r), \hat{\Psi}^\dagger(r)] = \delta(0) = +\infty$. If we interpretate this divergence physically, it comes from the fact that, in the Wigner picture, to take into account of quantum noise, some noise is added in each mode of the classical field $\Psi(r)$. Resultant for the infinite number of modes, it turns out to be infinity. That is why in equation (3.1) the average is taken which shows the discretization of the problem and makes suitable for numerical analysis.

3.1.2 TWA for a steady state polariton condensate

TWA extended formalism for polariton fluid was developed in 2005 in Ref [11] [13]. As we know in case of polariton, we use external pump and consider the loss terms, due to these two factors, stochastic equations eliminate the inaccuracy from TWA at some extend as long as interaction between polaritons are significantly weak. If, for numerical calculation the spatial resolution of real space spacing is Δx, the weak interaction limit can be expressed as $|g_{LP}| \ll \lambda \Delta x$. Because of this limit, we can not use the same formalism for strongly correlated polariton states. It works very well to investigate the quantum hydrodynamic effects such as analogue hawking process which is the outcome of the collective dynamics of a very large number of polariton condensate. (seems admissible so far).

Further we recall the moments of the Wigner function associated with any observable is given as [12],

$$\langle O_1 \dots O_N \rangle_W = \frac{1}{N!} \sum_p Tr[\hat{O}_{P(1)} \dots \hat{O}_{P(N)}] \tag{3.2}$$

Where the sum is carried over all the possible permutation of N objects. O_i represents ψ classical field at a random nodes of grid, while $\hat{O}_i$ is the corresponding quantum field operator. Approximated continuous field inside the cavity evolve according to the Hamiltonian of the system under GPE. For numerical analysis, one of the simplest way is to get the density matrix of a quantum system and applying the master equation to study its time dependent dynamics. Originally the master equation for a quantum system is motivated from the classical mechanics. For any quantum system, the equation of motion with density operator ρ is given as,

$$\dot{\hat{\rho}} = -i \left[\hat{H}, \hat{\rho} \right] \tag{3.3}$$

If one trace over all degrees of freedom except which are related to the lower polariton branch, a complete density operator ρ yields $\hat{\rho}_{LP}$. The dynamic of $W\{\psi\}$ is strictly equivalent to that of the quantum field. A direct numerical simulation of $\hat{\rho}_{LP}$ is not possible, but it can be achieved by a truncation of its Wigner quasi-distribution [14]. To evolve a system numerically with respect to time one has to work on discrete set of coordinates. TWA works on the same principle which gives a perfect discretized coordinate to the continuous space. Let $X = X_i$ be a random variables satisfying an Itô stochastic differential equation of the form

$$dX_i = F_i(x,t)dt + dW_i(t) \quad , \quad i = 1, ..., N \tag{3.4}$$

with $dW_i(t)$ a white noise. Let P be the probability distribution of X, then there is an equation, the Fokker-Planck equation, satisfied by P and equivalent to the Itô equation

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(F_i(x,t)P(x,t)) + \frac{1}{2}\sum_{i=1}^{N} \frac{\partial}{\partial x_i \partial x_j}(D_{ij}(x,t)P(x,t)) \tag{3.5}$$

provided that D is positive definite. The equivalence between these two equations is the principal idea for the truncated Wigner approximation, in our case the degrees of freedom are first continuous and become finite after discretization. From the master equation (3.4), one can derive the Fokker-Planck equation specific to the our system [14]. However in order to solve this equation numerically one needs to cast this Fokker-Planck equation to an Itô stochastic equation. This is only possible [14] if we truncates the higher order terms in the Fokker-Planck equation which has form of equation (3.6). Provided $\gamma_{LP} \gg g_{LP}/\Delta V$, where ΔV is the volume step of the simulation grid. Then the Itô equation for polariton system has the form [5],

$$\begin{aligned} d\psi(r,t) = &-i\left[\omega_{LP}^0 - \frac{\hbar}{2m_{LP}}\nabla^2 + V_{LP}(r,t) + g_{LP}\left(|\psi(r,t)|^2 - \frac{1}{\Delta V}\right) - \frac{i\gamma_{LP}}{2}\psi(r,t)\right]dt \\ &+ i\eta_{LP}E(r,t)dt + \sqrt{\frac{\gamma_{LP}}{4\Delta V}}dW(r,t) \end{aligned} \tag{3.6}$$

Where $dW(r,t)$ is a zero-mean, complex white noise satisfying [11]

$$dW(r,t)dW(r',t) = 0 \tag{3.7}$$

$$dW^*(r,t)dW(r',t) = 2dt\delta_{r,r'} \tag{3.8}$$

The parameter of the Wigner function thus verifies a Gross-Pitaevskii equation with additional white noise. A complete and rigorous derivation of the truncated Wigner approximation can be found here [14]. For now, ψ is only a continuous parameter associated with the Wigner distribution.
In our case, we use TWA for a given time step, and after a short period of time, field ψ reaches a steady state. Once system reached to the steady state, it start sampling. The system under study is considered as 1-dimensional. From equation (3.2), Wigner distribution for the two point diagonal function can be written as,

$$\langle\psi^*(x,t)\psi(x,t)\rangle_W - \frac{\delta_{x,x'}}{2\Delta V} = |\Psi_{LP}(x,t)|^2 + \langle\delta\hat{\Psi}^\dagger(x,t)\delta\hat{\Psi}(x,t)\rangle \tag{3.9}$$

Thus, one can interpret the Wigner realizations as a classical field with stochastic noise accounting for the quantum fluctuation of the field. When applying the truncated Wigner approximation to a black hole configuration (i.e. a configuration where the sound speed and fluid velocity crosses), we are interested in computing the two point diagonal correlation function or two-body density correlation. The truncated Wigner approximation for two points yields as,

$$\langle\psi^*(x)\psi^*(x')\psi(x')\psi(x)\rangle_W = \frac{1}{4!}\sum_{P\in S}\langle P\left(\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x')\hat{\Psi}(x)\right)\rangle \tag{3.10}$$

Moreover, knowing the commutation relations of the field $[\Psi_{LP}(x), \Psi^\dagger_{LP}(x')] = \delta(x-x') = \delta_{x,x'}/\Delta_V$, the two-body diagonal correlation function can be rewritten as, [13]

$$\begin{aligned} G^{(2)}(x,x') = \langle\hat{\Psi}^\dagger(x)\hat{\Psi}^\dagger(x')\hat{\Psi}(x')\hat{\Psi}(x)\rangle = \langle\psi^*(x)\psi^*(x')\psi(x')\psi(x)\rangle_W \\ -\frac{1}{\Delta V}(1+\delta_{x,x'}) \times (\langle\psi^*(x)\psi(x)\rangle_W + \langle\psi^*(x')\psi(x')\rangle_W - \frac{1}{2\Delta V}) \end{aligned} \tag{3.11}$$

We are interested in the correlation function normalized by the population which is expressed as equation (2.16). In condensate via Bogoliubov picture, we have $\hat{\Psi} = \Psi_{LP} + \delta\hat{\Psi}$, then the normalized correlation function of the quantum fluctuation can be viewed as $g^{(2)}(x,x') - 1$ [15].

3.2 Numerical results

3.2.1 Homogeneous profile and it's bi-stability

Homogeneous profile allows us to solve it analytically, thus by performing calculation and over the same homogeneous profile, one can get the idea of numerical results and its efficacy. In homogeneous case, the pump is homogeneous, it means $E = cst$, and the polariton potential is zero, $V_{LP} = 0$ and GPE equation (2.17) reads as,

$$\begin{aligned} i\partial_t \Psi_{LP}(x,t) = &\left[\omega_{LP}^0 - \frac{\hbar}{2m_{LP}}\nabla^2\right]\Psi_{LP}(x,t) + g_{LP}|\Psi_{LP}(x,t)|^2\Psi_{LP}(x,t) \\ &- \frac{i\gamma_{LP}}{2}\Psi_{LP}(x,t) + i\eta_{LP}E. \end{aligned} \tag{3.12}$$

The potential and pump being time independent, equation (3.12) reads,

$$\begin{aligned} i\partial_t \Psi_{LP}(x) = &\left[\omega_{LP}^0 - \frac{\hbar}{2m_{LP}}\nabla^2\right]\Psi_{LP}(x) + g_{LP}|\Psi_{LP}(x)|^2\Psi_{LP}(x) \\ &- \frac{i\gamma_{LP}}{2}\Psi_{LP}(x) + i\eta_{LP}E. \end{aligned} \tag{3.13}$$

Spatial homogeneity of the system implies a spatially homogeneous field $\Psi_{LP} = \Psi_0 = cst$, thus equation (3.13) becomes

$$[\omega_p - \omega_{LP}^0 - g_{LP}|\Psi_0|^2 + i\gamma_{LP}/2]\Psi_0 = i\eta_{LP}E \tag{3.14}$$

Taking the module of equation (3.14), we obtain as following,

$$[(\omega_p - \omega_{LP}^0 - g_{LP}|\Psi_0|^2)^2 + \gamma_{LP}^2/4]|\Psi_0|^2 = \eta_{LP}|E|^2 \tag{3.15}$$

Thus for a given pump intensity, we can solve equation 3.15 to find possible values of $|\Psi_0|^2$. Plotting the values of the sound speed (proportional to the module of the field) as a function of the pump amplitude, we find the solutions.

Figure 3.1 represents the analytical solution for equation (3.15), where the first turning point from the top is known as point C. The Bogoliubov theory is valid near point C of the bi-stability loop[5]. Solution of a homogeneous profile is easily obtained via finding the point c and corresponding pump amplitude. Once we have pump amplitude, we can call out the TWA numerical algorithm to get the quantum fluctuation over top of the condensate fluid. Further once

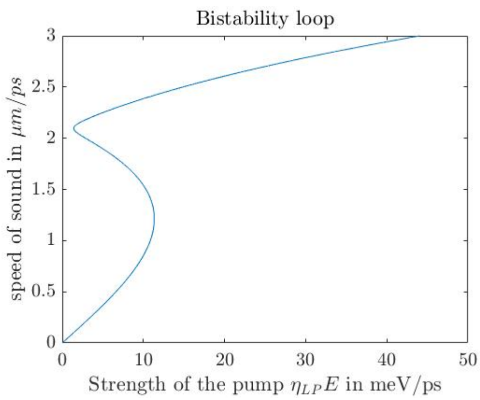

Figure 3.1: Bi-stability loop for homogeneous profile

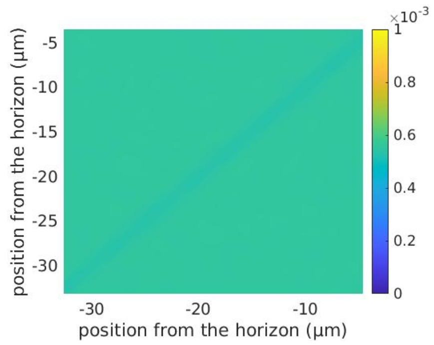

Figure 3.2: $g^{(2)}(x, x') - 1$ for homogeneous profile

we have the normalize correlation function $g^{(2)}(x, x')$, we can generate a visual representation of the same shown in figure (3.2), As we can see there is only one diagonal strip, which means there is no correlation between two points at the same time for homogeneous case.

3.2.2 In-homogeneous profile with horizon and spatial correlations

This section summarizes numerical results of the Hawking radiation emerging at horizon on an acoustic black hole. To make it more clear, recall that emission of Hawking radiation at the sonic horizon is outcome of inter-conversion of zero-point quantum fluctuations into observable radiation defined via Bogoliubov dispersion. To realize this we have to take the full GPE (2.17) with all the effects and solve it using TWA. Finally get the spatial density correlation function caused by quantum fluctuation over top of the fluid. Numerical plots of the normalized two point density correlation function $g^{(2)}(x, x') - 1$ is shown in figure (3.3).

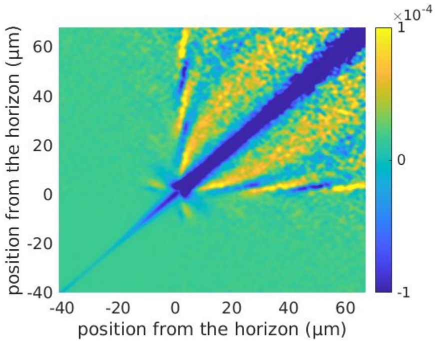

Figure 3.3: $g^{(2)}(x, x') - 1$ in presence of horizon for $800\mu m$ cavity size

Figure (3.3) shows numerous features which can be described as a direct outcome of the analogue Hawking process. The horizon position is calibrated as zero, where the two Bogoliubov excitations emerging as the Hawking pairs. At the horizon we can observe a weak correlation pattern between Hawking and its partner phonon, emitted in two opposite side upstream and downstream of the horizon. After a few micrometer, the correlation is disappeared, which represents the finite lifetime of the polariton condensate and dissipation effects.
In addition to the weak correlation pattern at the horizon, we can also observe a strong positive correlation in the downstream regime $x, x' > 0$, located parallel to the main diagonal strip. These intense density correlation pattern explains the correlation between partner and companion. Such numerical studies have been already presented by available litterateurs [16][17]. Still, other complex pattern and increasing width of main diagonal strip in downstream is not fully explained. In the same line, polariton condensate fluid can be explained in more better way via comparing the experimental results and numerical prediction. The normalize correlation function plots also enable us to get the density operator to study the quantum features of it, explained in the next chapter.

Chapter 4

Bipartite entanglement in polariton acoustic black hole

Objectives to study the entanglement in Hawking process with Bose-Einstein condensate acoustic black hole have been widely presented [18][19][20]. Goal to establish the bipartite entanglement formulation in polariton acoustic black hole and verify the same using quantities accessible with experimental data is the ultimate outcome of this project.

Till now in the preceding chapters, we build the basics theory and numerical study on spatial correlation in polariton condensates. This chapter summarizes the idea of studying entanglement in analogue gravity experiments, originally formulations are adopted from the BEC based studies [21][22]. The quantum systems such as BEC and polariton acoustic black hole enables us to mirror the similar phenomenon's formalism occurring in real astrophysical black holes. At the acoustic horizon, due to quantum fluctuation, in-going sounds like excitations emerge and scatter into outgoing excitations. These outgoing field modes travel in the both side of the horizon and induces the density correlations. And these correlations gives an indirect signature of the analogous hawking process. This effect was proven experimentally first time by the team of researchers from Trento and Bolgona [23].

To study the Bipartite entanglement, we first present the Bogoliubov transformation which connects the incoming modes with outgoing modes which are the key factors emerging from elementary excitations. We discuss a few other tools such as covariance matrix and Positive partial transpose criterion to measure the inseparability between three Gaussian modes [24].

4.1 Bogoliubov Transformation

As we learned from previous discussion, quantum fluctuation in moving polariton condensate results in the emission of quasi-particles which act like the hawking emission. In the far upstream and downstream regime, these excitations are characterized by the plane waves in two regime separated by sonic horizon. The elementary excitations modes form the basis set for the quantum fluctuation term $\delta \hat{|psi}$ with the Bogoliubov dispersion relation expressed in the last chapter. If V_u and V_d are the upstream and downstream velocities of the fluid, and c_α denotes the speed of sound which is defined as $c_\alpha = \sqrt{gn_\alpha/m}$ with $\alpha = u, d$ for upstream and downstream region respectively. We can define a new quantity called Mach number as,

$$m_\alpha = \frac{V_\alpha}{c_\alpha} \tag{4.1}$$

The fluid flow in two different region of upstream and downstream are subsonic $i.e., = m_u < 1$ and supersonic $i.e., = m_d > 1$ respectively. In the upstream region, the spectrum has two branches labeled as $0|in$ and $0|out$. In the downstream region there are four branches labeled as $1|in$, $1|out$, $2|in$ and $2|out$, this idea is adopted from the fundamental work on entanglement monotones in BEC [20]. In the analogue event horizon, there is a specific scattering process of elementary excitons can be observed. These scattering process are corresponds to the same in-going sounds like excitations scatter into outgoing excitations near sonic horizon. To build the mathematics further, we need to define the amplitude of these scattering process. For example, in-going mode identified as $1|in$ is transmitted to the other side of horizon along $0|out$ channel and some part reflected back along to the $1|out$ and $2|out$ channel, the transmission and reflection amplitudes are written as $S_{10}(\omega), S_{11}(\omega)$ and $S_{12}(\omega)$ respectively. A quantum operator corresponding to this whole process is denoted as $\hat{b}_1(\omega)$.

Similarly the scattering amplitudes for in-going incident mode along the $0|in$ and $2|in$ are denoted as $S_{01}(\omega), S_{02}(\omega), S_{00}(\omega)$ and $S_{20}(\omega), S_{21}(\omega), S_{22}(\omega)$ respectively. While the associated operator are written as $\hat{b}_0(w)$ correspond to 0 mode and $\hat{b}_2^\dagger(w)$ for modes along channel 2. These quantum operator also known as creation and annihilation operators follows the usual canonical commutation relations as expressed in equation (2.11). Other essential piece of information is that 3×3 scattering matrix $S(\omega)$ with elements $S_{ij}(\omega)$ follows the skew unitary relation [20], which is written as,

$$S^\dagger \eta S = \eta = S \eta S^\dagger \tag{4.2}$$

In more general description, operators denoted with b are corresponding to the scattering process initiated by single wave incident along *in* channels directed towards the horizon, thus operators b are associated with incoming modes. Similarly, we can also work on corresponding outgoing modes denoted by $\hat{c}_0(\omega), \hat{c}_1(\omega)$ and $\hat{c}_0(\omega)$. These outgoing operators are related with incoming operators via [20],

$$\begin{pmatrix} \hat{c}_0(\omega) \\ \hat{c}_1(\omega) \\ \hat{c}_2^\dagger(\omega) \end{pmatrix} = \begin{pmatrix} S_{00} & S_{01} & S_{02} \\ S_{10} & S_{11} & S_{12} \\ S_{20} & S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \hat{b}_0(\omega) \\ \hat{b}_1(\omega) \\ \hat{b}_2^\dagger(\omega) \end{pmatrix} \tag{4.3}$$

The definition expressed by equation (4.3) and (4.2) ensures that $\hat{c}$ follows the same relation as operators $\hat{b}$ do. Thus operators $\hat{c}$ also describes the bosonic polaritons. Thus the expression given by equation (4.3), provide the relation between incoming operators and outgoing operators which is also known as Bogoliubov transformation. For more detailed mathematical construction Ref [20][19] can be referred. Here, it is important to mention that all the scattering amplitude arranged as equation (4.3) depend on the energy $\hbar\omega$.

4.2 Measuring Entanglement

In this section, we will discuss the possibility to measure the entanglement among Gaussian modes analogue of Hawking process. In proceeding subsection, we have defined a basic formalism of Gaussian states and covariance matrix. The covariance matrix enables us to characterize the entanglement via symplectic eigenvalues of it. Further, in other subsection we define the positive partial transpose and schemes to quantify the degree of entanglement.

4.2.1 Gaussian states and covariance matrix

By algebraic definition, Quantum states given by,

$$\hat{\rho} = \Sigma_i p_i |\psi_i\rangle\langle\psi_i| \tag{4.4}$$

are the Gaussian states if and only if its associated characteristic Wigner function

$$\chi_w(\lambda) = Tr[\hat{\rho}\hat{D}(\lambda)] \tag{4.5}$$

is a Gaussian function. Where, $\lambda = (\lambda_1, ..., \lambda_N, -\lambda_1^*, ..., -\lambda_N^*)^T$, with $\lambda_i \in \mathbf{C}$, and $\hat{D}(\lambda)$ is a displacement unitary operator defined as,

$$\hat{D}(\lambda) = e^{\boldsymbol{\lambda}\mathbf{b}} \tag{4.6}$$

with $b = (\hat{b}_1, ..., \hat{b}_N, \hat{b}_1^\dagger, ..., \hat{b}_N^\dagger)^T$. In our case, the Bogoliubov transformation preserving the Gaussian nature, transform the incoming Gaussian state into the outgoing Gaussian state as expressed in equation (4.3). And it leads to the three-mode Gaussian states. To construct the covariance matrix, first let us introduce a vector which consists of quadrature operators $\hat{q}_i$ and $\hat{p}_i$ expressed as,

$$\xi = \sqrt{2}(\hat{q}_1, \hat{p}_1, ..., \hat{q}_N, \hat{p}_N) \tag{4.7}$$

Here the term quadrature loosely refers to the position $\hat{q}_i$ and momentum $\hat{p}_i$ operators corresponds to the respective mode i, and expressed in terms of creation and annihilation operators as,

$$\hat{q}_i = \frac{\hat{b}_i + \hat{b}_i^\dagger}{\sqrt{2}}, \hat{p}_i = \frac{\hat{b}_i - \hat{b}_i^\dagger}{\sqrt{2}i} \tag{4.8}$$

Now we define the covariance matrix σ_{ij} for a Gaussian state ρ as the real symmetric positive definite matrix [Real symmetric positive matrix refers to any Hermitian matrix with positive eigenvalues], expressed as,

$$\sigma_{i,j} = \frac{1}{2}\langle \hat{\xi}_i \hat{\xi}_j + \hat{\xi}_j \hat{\xi}_i \rangle - \langle \hat{\xi}_i \rangle \langle \hat{\xi}_j \rangle \tag{4.9}$$

Where ξ follows the commutation relation $[\xi_i, \xi_j] = 2i\Omega_{ij}$ with

$$\Omega = -\frac{i}{2} U \tilde{\Omega} U^T = \bigoplus_1^N \omega, \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.10}$$

In the expression (4.9), the averages $\langle ... \rangle$ are taken over the density matrix ρ which describes the state of the system. In general case ρ is expressed as (4.4). Entanglement properties of a quantum state defined as a Gaussian state are unchanged by local unitary transformation, so that the average values of position and momentum operators can be set to 0 [20]. Then a Gaussian state can be rewritten by its covariance matrix σ in form of 2×2 squares σ_{ij} as,

$$\sigma_{ij} = \begin{pmatrix} \sigma_i & \varepsilon_{ij} \\ \varepsilon_{ij}^T & \sigma_j \end{pmatrix} \tag{4.11}$$

with,

$$\varepsilon_{ij} = 2 \begin{pmatrix} \langle \hat{q}_i \hat{q}_j \rangle & \langle \hat{q}_i \hat{p}_j \rangle \\ \langle \hat{p}_i \hat{q}_j \rangle & \langle \hat{p}_i \hat{p}_j \rangle \end{pmatrix} \tag{4.12}$$

and,

$$\sigma_i = \begin{pmatrix} \langle 2\hat{q}_i^2 \rangle & \langle \{\hat{q}_i \hat{p}_i\} \rangle \\ \langle \{\hat{q}_i \hat{p}_i\} \rangle & \langle 2\hat{p}_i^2 \rangle \end{pmatrix} \tag{4.13}$$

where {...} is sign for anti-commutator. Due to the canonical commutation relations and positive density matrix condition, σ holds the following inequality,

$$\sigma + i\Omega \geq 0, \tag{4.14}$$

It is important to remind that the covariance matrix σ characterizes the entanglement of the corresponding Gaussian states. If it so, then definitely $\sigma' s$ eigenvalues carrying the information about the entanglement. In order to get the eigenvalues of σ matrix, we have Williamson's theorem given for any real symmetric positive-definite $2N \times 2N$ matrix σ. In this report the development is not being presented of how covariance matrix is a type of a real symmetric positive definite matrix and its symplectic transform, for the same Ref [20] [25] would be helpful. For the moment it is enough to state that for a matrix σ, there exists S such that the so called symplectic transform of σ by S has the canonical form unique up to the order of the v_j,

$$v = S\sigma S^T = diag(v_1, v_1, ..., v_N, v_N), \tag{4.15}$$

with $v_j \in \mathbb{R}$. The quantities $\{v_j\}$ presents the eigenvalues of σ and they can be computed by diagonalizing the matrix $|i\Omega\sigma|$, where Ω is as defined in equation (4.10). Further, these symplectic eigenvalues have a very significant physical meaning, they can be related with the mean particle number of a thermal state. By the definition of a single mode thermal state of a quantum system, whose density matrix in the Fock space spanned by vectors $|n\rangle$ is expressed as,

$$\rho^{th}(a) = \frac{2}{a+1}\Sigma_{n=0}^{\infty}(\frac{a-1}{a+1})^n |n\rangle\langle n|, \tag{4.16}$$

for now let say a is just an arbitrary parameter. If $\hat{n}$ denotes the number operator, then the parameter a has a simple relation with the mean particle number as $a = 2\bar{n}+1$, with $\bar{n} \equiv \langle \bar{n} \rangle = tr(\rho^{th}\hat{n}) = \frac{1}{2}(a-1)$. The beauty of the mathematical coincidence is that, $\rho^{th}(a)$ is a Gaussian state with 2×2 covariance matrix $\sigma^{th} = a\mathbb{I}_2$. It tells that a single mode covariance matrix in diagonal form explains a thermal state with mean particle number $\bar{n}$ and symplectic eigenvalues $v = a$. As we know the best way to have the idea of purity of any state is to find out the value $tr[\rho^{th}(a)]^2$, which reads $1/a$ for equation (4.16). In more straight way, the quantity a, which is the inverse of the purity of a single-mode reduced density matrix, represents as the *localmixedness* [20]. From a very simple system of Harmonic Oscillator, we are familiar with the number operator whose expectation value represents the mean particle number as $\bar{n}_i = \langle \hat{c}_i^\dagger \hat{c}_i \rangle$, which is related with the quantity of local mixedness a.

Note that, via definition of quadrature operator and Bogoliubov transformation, equation (4.12) & (4.13) simply defined as following [22],

$$\sigma_i = (1 + 2\langle \hat{c}_i^\dagger \hat{c}_i \rangle)\mathbb{I}_2 \tag{4.17}$$

and

$$\begin{aligned} \varepsilon_{ij} &= 2|\langle \hat{c}_i \hat{c}_j^\dagger \rangle|\mathbb{I}_2, \ [i, j = 0, 1] \\ \varepsilon_{i2} &= 2|\langle \hat{c}_i \hat{c}_2^\dagger \rangle|\sigma_z, \ [i = 0, 1] \end{aligned} \tag{4.18}$$

From preceding sections, we recall that the parameters associated with the outgoing channels are $\hat{c}_i$ that depend on the energy $\hbar\omega$ of elementary excitations. So this way the quantity a should also depend on ω. From ref [20], mathematical development suggest that parameters a_i can be written as a function of the scattering matrix coefficients as,

$$\begin{aligned} a_0(\omega) &= 1 + 2|S_{02}(\omega)|^2 \\ a_1(\omega) &= 1 + 2|S_{12}(\omega)|^2 \\ a_2(\omega) &= -1 + 2|S_{22}(\omega)|^2 \end{aligned} \tag{4.19}$$

It was a quite interesting journey to connect the scattering coefficient with the parameter a which is in fact not directly related with the phenomena. To sum up the above dissuasion, from any acoustic black hole configuration, we can calculate the scattering amplitudes $S_{ij}(\omega)$, which connects us with the three local mixedness associated with the respective modes. In the next section we will understand that how these mixedness quantities can explain the degree of entanglement. These scattering coefficient are accessible via experimental data.

4.2.2 PPT criterion and bipartite entanglement

The positive partial transpose criterion also known as Peres-Horodecki criterion decides the separability of mixed state where the general Schmidt decomposition does not applicable such as continuous Gaussian states [26]. In our case total number of modes are three (i, j, k). To investigate bipartite entanglement between any two-modes, we can obtain the reduced density matrix by tracing out the third one. For instance we can obtain the covariance matrix σ_{ij} (4.11) associated with two modes $i.j$ by tracing out the mode k. And its symplectic eigenvalues will be

$v_{\pm}$. Quantitatively according to the PPT, the necessary and sufficient separability criterion can be set as the positivity of ρ^{PT} which is equivalent to the positivity of symplectic eigen values denoted by v_j^{PT},

$$v_j^{PT} \geq 1, \tag{4.20}$$

The symplectic eigen values v_{ij} of the reduced covariance matrix 4.11 is given by,

$$2(v_{\pm}^{PT})^2 = \Delta_{ij}^{PT} \pm \sqrt{(\Delta_{ij}^{PT})^2 - 4det\sigma_{ij}} \tag{4.21}$$

with $\Delta_{ij}^{PT} = det\sigma_i + det\sigma_i - 2det\varepsilon_{ij}$.

For two mode state, the PPT criterion is equivalent to the condition $v_{-}^{PT} \geq 1$ only, because the other eigenvalue with positive sign is always greater than 1 [26]. The local mixedness defined in the equation (4.19), depend on the frequency ω, and from above relation its fair to say that the lowest possible symplectic eigenvalue v_{-}^{PT} also depends on ω. Since $a_i \geq 1$, one can say that for $0|1$ mode, $v_{-}^{PT} \geq 1$. It means the reduced state of modes $0|1$ is always separable. If we connect the terms from Hawking process, the Hawking and the companion are not entangled. From the calculation done by Isorod et al for the BEC system, they found that the eigenvalues v_{-}^{PT} of the reduced covariance matrices σ_{02} and σ_{12} are lower than 1. Which signify that the reduced state of modes $0|2$ and $1|2$ also known as Hawking-partner and Companion-partner respectively, are entangled for the certain range of the frequency ω.

For more simplicity, we can say that with the PPT criterion the bipartite states are entangled if and only if,

$$1 - v_{-}^{PT} > 0 \tag{4.22}$$

Our prime candidate is Hawking-partner pair $0|2$. We can also compare the PPT's results with the Cauchy-Schwarz (CS) criterion. According to CS criterion, the pair $0|2$ is entangled if and only if,

$$\Delta_{CS} \equiv |\langle \hat{c}_0 \hat{c}_2 \rangle_{th}|^2 - \frac{(a_{0,th} - 1)(a_{2,th} - 1)}{4} > 0 \tag{4.23}$$

4.3 Analytical calculation of S_{ij} and entanglement detection

The first expression for each scattering modes in analogue black hole configuration was given by Larre et al. [27]. It was derived for the black hole horizons in

Bose-Einstein condensates but here we assume that for similar waterfall configuration, it also holds true for the Polariton based system. For the low ω limit the coefficients of S-matrix are expressed as,

$$\begin{aligned} S_{i,0} &= F_{i,0} + \mathcal{O}(\varsigma) \\ S_{i,1} &= F_{i,1}\sqrt{\tfrac{1}{\varsigma}} + \mathcal{O}(\varsigma^{1/2}) \\ S_{i,2} &= F_{i,2}\sqrt{\tfrac{1}{\varsigma}} + \mathcal{O}(\varsigma^{1/2}) \end{aligned} \tag{4.24}$$

where $\varsigma = \hbar\omega/gn_u$ and $F_{i,j}$ are dimensionless complex coefficient. $F_{i,j}$ are determined analytical for different black hole configuration which are only dependent on the Mach number in upstream and downstream region. In case of waterfall configuration $F_{i,j}$ are expressed as [27],

$$\begin{aligned} |F_{02}|^2 &= \tfrac{2m_u(1-m_u)^{3/2}(1+m_u^2)^{3/2}}{(1+m_u)^{1/2}(1+m_u+m_u^2)^2} \\ |F_{12}|^2 &= \tfrac{(1-m_u)^{7/2}(1+m_u^2)^{3/2}}{2(1+m_u)^{1/2}(1+m_u+m_u^2)^2} \\ |F_{22}|^2 &= \tfrac{(1-m_u^4)^{3/2}}{2(1+m_u+m_u^2)^2} \end{aligned} \tag{4.25}$$

where m_u is defined in the equation (4.1) as the upstream Mach number. Using the above expression we can calculate the scattering amplitude corresponding to the different propagating modes. Once we have scattering amplitudes it is straight forward to get the values of local mixedness of each three outgoing norms. If we assume the temperature at $0k$, the mean particle number and cross term are calculated using following formulae,

$$\begin{aligned} \langle \hat{c}_i \hat{c}_2 \rangle &= S_{i0}S_{20}^* + S_{i1}S_{21}^* = S_{i2}S_{22}^* \\ \langle \hat{c}_i^\dagger \hat{c}_i \rangle &= |S_{i2}|^2 \end{aligned} \tag{4.26}$$

where $i = 0, 1$. For more detailed derivation of the above expressions refer [27] and [22].

Figure 4.1 summaries the variation of local mixedness a_i for each mode with respect to the frequency. a_i are calculated analytically with $m_u = 0.79$ considering a waterfall configuration. For all three modes, a_i behave constantly after maximum limit of ω, which is 1.4 in our case. The maximum limit of the ω is estimated via dispersion spectra. If we compare this result with the BEC's system result for the same configuration [22], there is noticeable change in the local mixedness of mode 2, while other two modes behave similarly. In case of

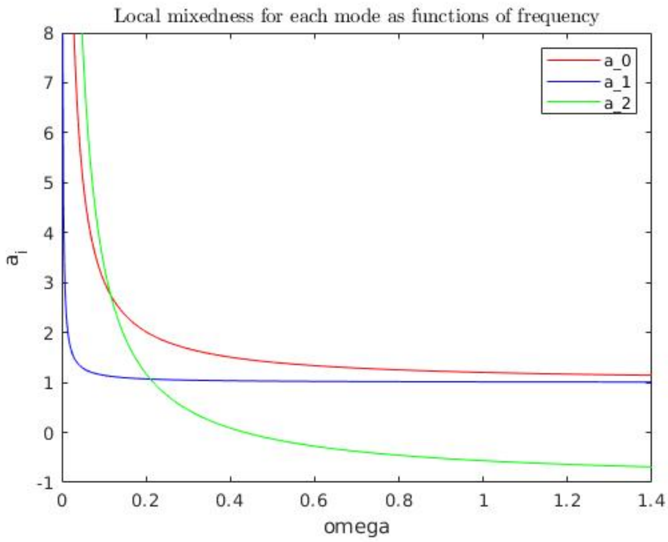

Figure 4.1: Local mixedness $a_i(\omega)$ for each mode 0, 1 and 2 as function of ω

BEC, under maximum limit of ω there is only one section between 0 and 1 mode, while in our case there is two sections between 0-2 and 1-2. The local mixedness a_0 and a_1 go to 1 when ω reaches to its maximum limit while a_2 become negative. In case of modes 0 and 1, its clear to say that once ω achieve its maximum value, their populations vanish. As per the preceding explanation, in our case of three-mode Gaussian state, to determine the entanglement content of any prescribed bipartition, the three symplectic invariant a_i are enough. Study is ongoing so it would be good to wait for final results. It is important to note that the variation in the local mixedness with respect to ω is restricted via the Mach number in upstream region, which is the key factor behind a perfect design of the experiment to get the significant trace of the entanglement among Hawking and partner pair of Hawking phenomena. To get the perfect ensembles of parameter it is recommended to get the best value of the Mach number in upstream region. We can study about variation of the bipartite entanglement with the Mach number in upstream region m_u. The best value of m_u would be corresponding to the maximal value of bipartite entanglement.

Figure 4.2 shows the evolution of Cauchy-Schwarz criterion for the analogue Hawking pairs as a function of ω. Plots is obtained for the upstream Mach number $m_u = 0.79$. If we compare with the BEC results, the measure of entanglement should vanish after the maximum value of ω but here they are not. Technically, beyond maximum limit of ω, the population of $d2$ *norm* vanishes. Rough observation tells that bipartite system $0|2$ should be entangled as far as the value of CS is positive. The measure do not violate at $\omega = 0$. This observation is

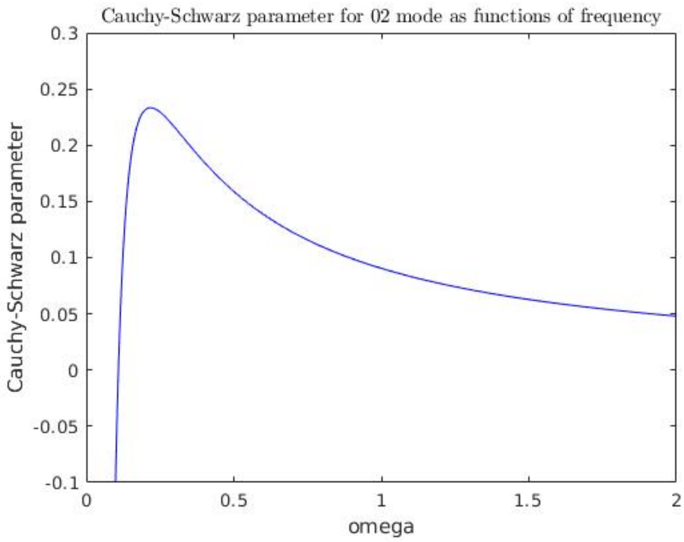

Figure 4.2: Variation in the Cauchy-Schwarz parameter with ω

based on the argument of presence of uniform condensate fluid connecting the both asymptotic regions (upstream and downstream) in the system. Mathematically the parameters depend up on the local mixedness and two particle spectra $\langle \hat{c}_0 \hat{c}_2 \rangle$. CS measure can give us the idea whether modes are non separable or not but via PPT we can also quantify the entanglement and tell the precise degree of entanglement. Measure of the logarithmic negativity is one of the best mathematical tool to study about the same. As per the physical observation and argument made above, CS measure should vanish once it reaches to the maximum frequency limit but it does not. Similar result was presented by Jeff for the BEC system [28]. Entanglement detection via means of density-density correlation traces could give us more clear picture in this regard. One thing is important to note that through experimental data we can access the density-density correlation and one particle spectra which directly connect us with the entanglement parameters.

4.4 Entanglement measurement by density-density correlation function with numerically simulated data

The idea is to use reciprocal space density-density correlation plot to obtain the local mixedness and correlation amplitude for a particular given frequency. Ana-

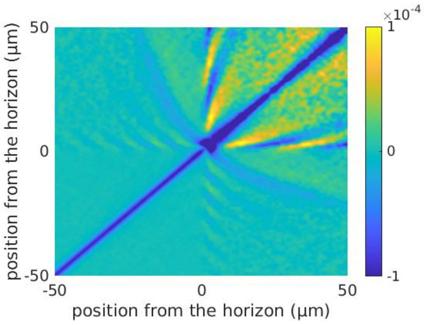

Figure 4.3: g(x,x')

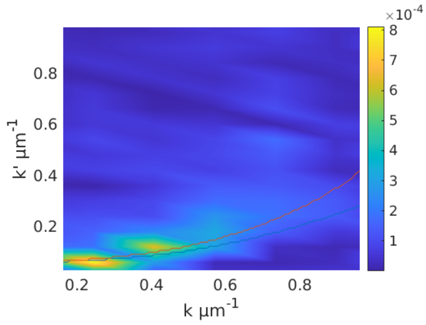

Figure 4.4: g(k,k'),ud_+

lytically its straight forward to calculate all the quantities but it won't fulfil the goal of studying Hawking process using an analogue sonic horizon experiment. Because from the experiment we can only access the density correlation profile and the fluctuation density in upstream and downstream regions. Here fluctuation density is also called as one particle spectra which is mathematically nothing but the n_i. Now how can we obtain the reciprocal density profile of the correlations? There is one very simple way of performing Fourier transform of real space density-density correlation plot of a certain section containing significant data.

Indeed the plots shown in figure (4.4) is being computed via Fourier transform of lower right section of the real space correlation plot shown in the figure (4.3). To be precise we can choose relevant window to compute the Fourier transform. Now the results are more pronounced in the reciprocal space which allows us to have the idea of correlation strength for a particular pair of (k, k'). And through the dispersion relation k directly connects us to specific frequency. Well, now our job become easy to get the entanglement but not very near. If we observe carefully the reciprocal correlation data, we see two separate trace matching with the analytical correlation trace obtained from the dispersion relation for corresponding pairs of k, k'. One, which is very bright, corresponds to our main attraction Hawking-Partner pair $u_{out} - d_{2out}$. There is one more trace with very low amplitude which is said to be the Hawking-companion pair $u_{out} - d_{1out}$. Lets go a bit further and try to understand mathematical construction of this Fourier operation. When we perform the said FFT of any section of a real space correlation data, we obtain following quantities,

$$\begin{aligned} \frac{\mathcal{S}_0^{-1}}{\sqrt{L_u L_d}} \int_{-L_u}^{0} dx \int_{0}^{L_d} dx' \exp -i(k_H x + k_P x') g^{(2)}(x, x') \\ = d(\omega)\langle \hat{C}_U(\omega)\hat{C}_{D2}(\omega)\rangle = d(\omega) S_{u,d2} S^*_{d2,d2} \end{aligned} \tag{4.27}$$

where, $d(\omega)$ is the correction factor which depends on the windowing for the FFT operation. $\mathcal{S}_0 = (u_{k_H} + v_{k_H})(u_{k_P} + v_{k_P})$ is known as the static structure factor, defined via Bogoliubov amplitudes of the Hawking and partner modes respectively. L_u *and* L_d are the size of the chosen rectangle or section in upstream and downstream region respectively. A precise calculation of the correction factor can be found in the ref [22]. It is defined as following,

$$d(\omega) = \frac{1}{\sqrt{a(\omega)}} \; if \; a(\omega) \geq 1, \; and \; d(\omega) = \sqrt{a(\omega)} \; if \; a(\omega) \leq 1 \\ with \; a(\omega) = \frac{L_d |V_g(k_H)|}{L_u V_g(k_p)} \tag{4.28}$$

where $V_g(k_H) = \partial\omega/\partial k$ is the group velocity of energy $\hbar\omega$. Similarly for the partner mode.

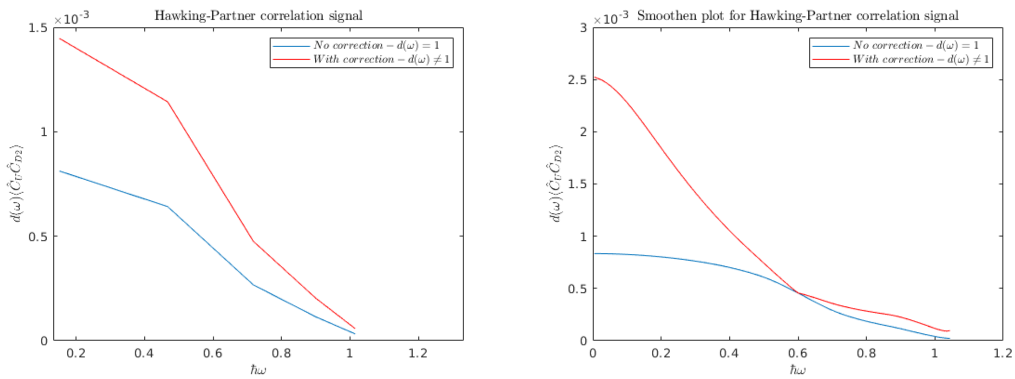

Figure 4.5: Hawking-partner correlation amplitudes, with and without increased data points [Right to Left]. Blue curve is representation of right hand side of the equation (4.27) with no correction, while red curve is obtained with the same equation considering proper corrections.

Figure (4.5) shows the Hawking-partner pair correlation's amplitude with respect to the energy $\hbar\omega$. Blue curve is computed from the equation (4.28) assuming that there is no correction which means $d(\omega) = 1, \; \forall\omega$. The red curve also computed using the equation (4.28) with $d(\omega) \neq 1$. As we know that group velocity is nothing but the slope of the dispersion plots for respective value of wave number. So, if we compute the quantity $a(\omega)$ as explained above, it turns out that there are values ≤ 1 and some of the values ≥ 1. During computation of the correction factor we have to be careful, as it changes for both the case. Precisely, if we can get the perfect correction factor considering proper windowing, it is

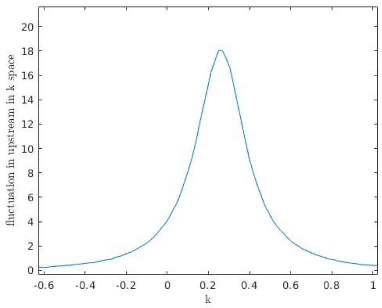

Figure 4.6: Upstream $\langle \hat{C}_u^\dagger \hat{C}_u \rangle$

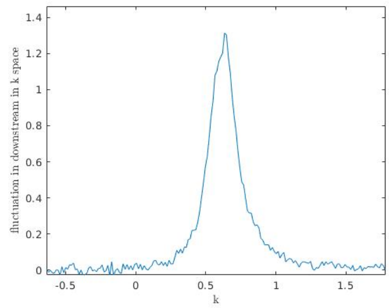

Figure 4.7: Downstream $\langle \hat{C}_d^\dagger \hat{C}_d \rangle$

possible to calculate all the scattering amplitudes as presented in the preceding section numerically.

We again recall the expression of local mixedness, the quantity is defined through the occupation number of the respective modes. There has been number of ways to compute the single particle spectra such as doing the FFT along a line of upstream region to get the occupation of U_{out} mode or to follow the analytical expression considering the thermal effect so that it can accommodate the finite temperature effects. In our case, we wanted to use the fluctuation density to compute the single particle spectra $\langle \hat{C}_i^\dagger \hat{C}_i \rangle$ in reciprocal space. We first obtain the spatial dependent wave function for the fluid via TWA and then performing FFT over all the realization with effective normalization we get the same wave function in reciprocal space. Now if we chose to perform windowed FFT over only upstream data, we can accurately derive the data for respective region. Once we have this data we can calculate the exact fluctuation density or single particle spectra for upstream and downstream part of the system as shown in the figure (4.6) and (4.7) respectively. To be sure if obtained data are giving the right result, we can compare the range of wave numbers obtained with the dispersion relation. Now yet problem is not fully solved. As we know in the upstream, there is only U_{out} mode, while in the downstream region, there are two modes of $D2_{out}\ andD1_{out}$, and the data we get from the analysis is mix of both the modes in downstream region. From the dispersion relation, we confirm that $D2_{out}\ andD1_{out}$ modes are negative and positive solutions in the downstream region respectively. Now the question is, can we separate these two norms by any medium of mathematical formulation? If we talk about the quantum fluctuations on top of the condensate explained in the second chapter, which can also be described by a quantum field $\delta\hat{\Psi}_k$ in reciprocal space. All the observable in real space can be computed directly with the truncated Wigner approximation. And via using FFT we can derive the expression and do the same calculation

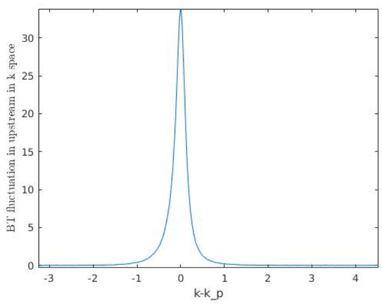

Figure 4.8: Upstream $\langle \hat{\phi}_u^\dagger \hat{\phi}_u \rangle$

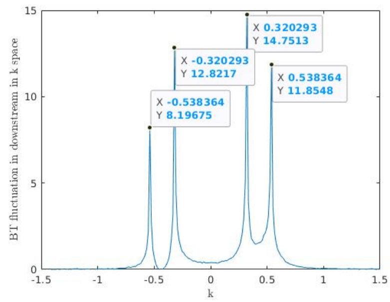

Figure 4.9: Downstream $\langle \hat{\phi}_d^\dagger \hat{\phi}_d \rangle$

in reciprocal space. The best part of working in the reciprocal space is that we have precise information of all the observable associated to an unique frequency. As we know that phonons are the excitation of the field, which can be obtained via performing the Bogoliubov transformation over the field of lower polariton. So the proposal is to perform Bogoliubov transformation with proper windowing over upstream and downstream part. Performing a Bogoliubov transformation on the fluctuation is done in [29]. The collective phonon destruction operator is given by [29],

$$\begin{aligned} \hat{\phi}(k) &= u_k \delta \hat{\Psi}_k + v_k \delta \hat{\Psi}^\dagger_{-k}, \\ with\ u_k &= \frac{\sqrt{\Omega_k + m_{LP} c_s^2} + \sqrt{\Omega_k - m_{LP} c_s^2}}{2(\Omega_k^2 - m_{LP}^2 c_s^4)}, \\ and\ v_k &= \frac{\sqrt{\Omega_k + m_{LP} c_s^2} - \sqrt{\Omega_k - m_{LP} c_s^2}}{2(\Omega_k^2 - m_{LP}^2 c_s^4)} \end{aligned} \tag{4.29}$$

where Ω_k is detuning coefficient defined as $\Omega_k = k^2/2m_{lp} - \omega_p + E_0 + 2m_{lp}c_s$ and Bogoliubov coefficients are normalised such as $|u_k|^2 - |v_{-k}|^2 = 1$. So, now argument is that two outgoing modes $D1_{out}\&D2_{out}$ are the positive (ω_+) and negative (ω_-) norms of the dispersion solution in the downstream region respectively. In equation (4.29), terms $\delta\hat{\Psi}_k$ and $\delta\hat{\Psi}{-k}$ represent the collective excitation plane wave of forward and backward propagation with $+k$ and $-k$ wave number individually. With respect to our system polaritons are the quasi-particles and under Bogoliubov transformation $u_k\&v_k$ are also known as quasi-particle amplitudes. Interesting part is that the collective phonon destruction operator defined in equation (4.29) is nothing but the superposition of plane wave moving in $+k$ and $-k$ direction. So if we apply this operator or conjugate of the destruction operator which is a creation operator, these operators will destroy or create a phonon in condensate which implies a decrease or increase of particle in both the direction $+k$ & $-k$.

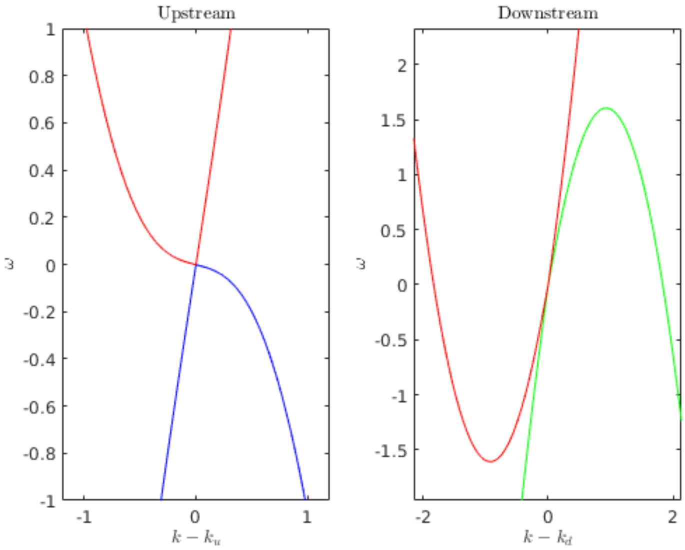

Figure 4.10: Dispersion relation of polariton flow in the upstream and downstream with respect to laboratory frame. The red curves are the positive norm of the dispersion solution while other blue and green curves are the negative norm in up and down stream respectively.

Well, now I think this much explanations are enough to understand the results shown in the figure (4.8) and (4.9) which is $\langle \hat{\phi}^\dagger_\alpha(k)\hat{\phi}_\alpha(k)\rangle$ calculated in reciprocal space for upstream and downstream region separately with proper windowing. In the upstream part, there is only one outgoing mode so it is expected to have one peak centralize at zero, while in the downstream region there are four peak at corresponding wave number. Wave number of each peaks are verified with dispersion results as shown in figure (4.10), which is obtained using relation expressed in section (2.2.4). Almost, we do have all the expected quantities to get the entanglement measurement and to quantify it. Still it might take a significant time to have the final results, soon we will see the concluding results and a nice article. Thank you for your suggestions to make this doc error free :)

Chapter 5
Conclusion

In this report, I presented a detailed review on analogue gravity and the experimental efforts done till now to realise the quantum effects of Hawking process. Order of this whole internship starting from 10th of May, first task was to get the numerical results for the homogeneous profile of the polariton condensate in absence of the defect, which is a key factor to generate the black hole configuration. In the second phase of the project, I extended the polariton system to study the Hawking process. The original code to study this process was provided by collaborator prof Iacopo. Code was run on matlab software with multiple configuration and the final best result achieved till now is presented in the chapter 3. The main aim of this project is to study the entanglement in Hawking process which is a proof of the quantum effect of Hawking process. Before getting to the entanglement part, it was necessary to understand the induced correlation and its intensity. The correlation can be classical and quantum. The maximally correlated states are considered under entanglement. Right now the experiment been performed at LKB Paris lab is not capable enough to study the quantum effects. To make a perfect experimental arrangement which can access the information at quantum level, it is necessary to do the same using numerical tools and develop the setup with numerically proven parameters. This is how studying entanglement for polariton based system become more important. As there are no literature's available for polariton fluid which can explain the entanglement, our approach is to use the proven formulation for BEC system and make changes according to the polariton fluid. The polariton based system is much accessible, stable and workable at the room temperature with the use of cryo equipment. While in the conventional BEC system, needed temperature is in the order of Nano-kelvin.

Final results about entanglement is yet to be established. As a preliminary results, I presented an acoustic black hole configuration in polariton fluid, density-density correlation in real and reciprocal space, variation in one & two particle spectra with respect to the energy $\hbar\omega$, Bogoliubov transformation and analytical results for CS measurement. In the region of sonic horizon, creation of elementary particle due to quantum fluctuation on the top of the polariton fluid are the signature of Hawking process in the real black hole. After completion of this project following results are expected,

- Numerically calculated local mixedness quantities $a_i(\omega)$ as a function of the dimensionless quantity $\hbar\omega/gn_u$.
- Measure of bipartite entanglement as a function of dimensionless quantity $\hbar\omega/gn_u$.
- Evolution of the PPT measure for the bipartite modes, specifically for the analogue Hawking pairs $(0|2)$.
- Explore the possibility of the precise experimental parameters to investigate the entanglement phenomena.

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